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**A Simplified Feedback Approach to Optimal  
Resource Management**

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# A Simplified Feedback Approach to Optimal Resource Management

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## **Abstract**

Optimal exploitation of a renewable capital stock is derived as a feedback rule for a general dynamic optimization problem with a single resource. By a feedback rule is meant that optimal exploitation is given as an explicit function of the capital stock. The value of the simplified approach will be appreciated by all who have tried to determine the separatrix solution numerically and experienced problems. The method described here can be applied to fisheries management, animal stock conservation and conservation of the environment in general. The operationality of the method outlined here is illustrated by two simple examples, one related to fisheries management and one related to pollution control.

## INTRODUCTION

Many, if not most, of the world's renewable resources are subject to excessive exploitation and should therefore be managed carefully. The reason for the excessive exploitation is the well-known tragedy of the commons (Scott, 1955; Hardin, 1968), and is often reinforced by unfortunate circumstances such as government's subsidies, etc. This is, for example, very often the case in the fisheries sector. This emphasizes the need for management based on realistic models that include both biological and economic aspects. Very often, however, such models are made either by biologists or by economists and therefore lack one of these aspects, biology or economy, but are quite sophisticated with respect to the other aspect. As an example can be mentioned bio-economic models based on intricate year-class models for population dynamics combined with very simple economics, that is constant prices and constant costs per unit harvest. Such models lead, correctly, to the conclusion that most rapid approach paths (MRAP) or bang-bang solutions are optimal. Although MRAPs have been shown to perform well for special cases (Clark, 1976), such paths are highly unmanageable and unrealistic in practice and are usually a result of oversimplification of the problem.

Another example is pollution models where the decay of pollution is either constant or linear whereas in reality the decay may be a very complex process. It has been shown that the time path of optimal carbon taxes is strongly dependent upon the shape of the decay function (Sandal and Steinshamn, 1998).

The aim of this article is to propose a simplified approach to adaptive management. The model can in principle be applied to many different types of renewable resources, but it is here exemplified with fisheries management and pollution control. Both the objective function and the dynamic constraint may be general functions (within the class of functions that give a unique solution) in the control variable as well as the

state variable. As will be seen, a certain degree of regularity of all expressions involved is required for the problem to be meaningful. The outcome of the model is an implicit closed-form expression for the optimal exploitation level of the resource given as a feedback rule. As such this article is a generalization of Sandal and Steinshamn (1997a). A feedback rule means that the optimal control is a function of the state variable. The value of closed-form expressions for the optimal control will be appreciated by all who have tried to find such rules numerically and experienced problems. Feedback rules also represent adaptive management as the control variable changes immediately when new knowledge about the state variable is available. The term adaptive, as it is used here, refers to Passively Adaptive Policies (PAP) as used by Walters (1986) and not to Actively Adaptive Policies (AAP). The feedback rule is PAP as it can absorb sporadic shocks and disturbances to the system, and the model changes as the data series grow. When we have closed-form solutions, the method also easily lends itself to AAP, but this has not been given emphasis in this paper. Instead emphasis has been put on finding a non-trivial closed-form feedback solution.

Further, closed-form expressions easily lend themselves to parameter analysis and comparative dynamics by differentiating. This is the case even when the expressions are implicit and the actual path has to be found numerically. It is, however, not the case when using conventional "brute force" numerical methods to find the optimal paths as suggested by e.g. Conrad and Clark (1987).

The difference between the model presented here and the model presented in Sandal and Steinshamn (1997a), which was based on a quadratic welfare function, is that with a quadratic function the optimal feedback control can be solved explicitly whereas with a general model the optimal control will usually be given implicitly. A version of the quadratic model including a general stochastic process in the biological submodel can be found in Sandal and Steinshamn (1997b). The generality of the method is revealed in Sandal and Steinshamn (1998) where a similar approach is applied to the

analysis of carbon taxes. The model can also easily be adopted to non-renewable resources.

In the following the optimal feedback rule is derived with and without discounting. With discounting perturbation methods are resorted to in order to find analytical expressions. Perturbation methods in resource problems have also been applied by other authors, e.g. Ludwig (1979) and Ludwig and Varah (1979) but not on the discount rate as it is here. For example, Ludwig (1979) perturbs on the stochastic term in a linear (bang-bang) model.

Finally, our model is illustrated by two examples to emphasize the operationality of this method.

## MODEL

The problem analyzed in this article is a quite general dynamic optimization problem that can be formulated

$$\max \int_0^{\infty} e^{-\delta t} \Pi(x, y) dt$$

subject to

$$\dot{x} = f(x, y), \quad \lim_{t \rightarrow \infty} x(t) = x^*.$$

It is assumed that the functions fulfill the Mangasarian sufficiency theorem for infinite horizon (Theorem 13 in Seierstad and Sydsæter (1987)). The variable  $x$  is the state variable, that is the size of the resource, and  $y$  is the control variable, that is the exploitation of the resource. The steady state for the resource is denoted  $x^*$ . Further,  $t$  denotes time,  $\delta$  is a constant discount rate and dots are used to denote time derivatives. All variables are in current values unless otherwise is stated explicitly. The time horizon can, of course, be finite, but for the management of renewable resources it is natural to emphasize an infinite horizon. By finite, in this connection, is meant

to find the same stationary solution in finite time. The complete finite time problem, with endogenous final state, usually implies mining the resource which is undesirable both from a practical and a political viewpoint.

The current value Hamiltonian for this problem is given by

$$\mathcal{H} = \mathcal{H}(x, y, m) = \Pi(x, y) + mf(x, y)$$

where  $m$  is the costate variable which in optimum is interpreted as the shadow value of the resource. The first-order conditions for the maximization problem are the usual

$$\begin{aligned} \mathcal{H}_y = 0, \quad \dot{x} &= f(x, y) \\ \dot{m} = \delta m - \mathcal{H}_x, \quad \dot{\mathcal{H}} &= \delta m \dot{x} \end{aligned} \tag{1}$$

The last of these equations follows from the three previous ones<sup>1</sup>. The maximum condition,  $\mathcal{H}_y = 0$ , implies  $\Pi_y + mf_y = 0$ . From this follows that the optimal shadow value of the resource can be written as a function:

$$m = M(x, y) = -\frac{\Pi_y}{f_y}. \tag{2}$$

As this is a known function when we know  $\Pi$  and  $f$ , it can be used to eliminate the shadow value. After the shadow value,  $m$ , has been eliminated, we get a new function which is always equal to the Hamiltonian in value along an optimal trajectory but it is different as a function. This new function can be defined as

$$P(x, y) = \mathcal{H}(x, y, M(x, y)). \tag{3}$$

Remembering the usual interpretation of the Hamiltonian as the rate of increase of total assets, it is natural to name the new function the total economic rent for short (Clark, 1990).

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<sup>1</sup> $\dot{\mathcal{H}} = \mathcal{H}_y \dot{y} + \mathcal{H}_x \dot{x} + \mathcal{H}_m \dot{m}$ . However,  $\mathcal{H}_y = 0$ ,  $\mathcal{H}_x = \delta m - \dot{m}$ , and  $\mathcal{H}_m = \dot{x}$ .

For the purposes in this paper, and resource management in general, it is extremely useful to be able to express the optimal control as a function of the state variable. This is referred to as a feedback rule and denoted  $y(x)$ . Inserting this function into (3) and differentiating with respect to time yields:

$$\dot{P} = \frac{dP}{dx}\dot{x} = \left( \frac{\partial P}{\partial x} + \frac{\partial P}{\partial y} \frac{dy}{dx} \right) \dot{x}.$$

By construction  $\dot{P} = \dot{\mathcal{H}}$ , and hence, by the last equation in (1), this yields a first-order differential equation that can be used to determine the feedback control. This equation is given by

$$\frac{dP}{dx} = \delta \cdot M(x, y), \quad (4)$$

and this is the basic equation that will be used in the following in order to derive optimal feedback control rules. The analysis is divided in two parts. First the case without discounting is investigated, then the case with discounting.

### **The feedback rule without discounting**

As has been shown in several studies, the positions of optimal paths are not very sensitive to changes in the discount rate as long as the discount rate is dominated by the maximum growth rate of the resource (Mendelssohn, 1982; Nordhaus, 1991; Sandal and Steinshamn, 1997a). This is the main reason for emphasizing the case without discounting. Another reason is that discounting itself puts a strain on the resource as it implies less emphasis on the future relative to the present. For completeness, however, the case with discounting will also be analyzed. The value of the objective function may, of course, be highly sensitive to changes in the discount rate.

Without discounting ( $\delta = 0$ ) it is immediately seen from (4) that the value of the Hamiltonian,  $P$ , is constant,  $P = P_0$ . This, combined with the expression for the costate variable given in (2), yields an operational feedback rule to determine optimal

harvest when we have decided the constant  $P_0$ . The feedback rule,  $y$  as a function of  $x$ , is given implicitly by

$$\Pi(x, y) + M(x, y)f(x, y) = P_0. \quad (5)$$

The only question that remains is how to determine the constant  $P_0$ . With an infinite time horizon  $P_0$  should be equal to the maximum value of the sustainable economic rent defined as

$$S(x) = \Pi(x, y)|_{f(x, y)=0}$$

which is a function of  $x$  alone, see Sandal and Steinshamn (1997a). This can be seen directly from (5) because in steady state the left-hand side of this equation reduces to  $S$ , and, as this represents net revenue, we want it to be as large as possible. For a shorter time horizon,  $T$ , the relationship between  $P_0$  and  $T$  is given by

$$T = \int_{x_0}^{x^*} \frac{dx}{f(x, y(x; P_0))}$$

where  $y(x; P_0)$  is the solution of (5).

The optimal steady state is given by  $x^* = \arg \max S(x)$ , and the corresponding exploitation level can be found by solving  $f(x^*, y) = 0$  with respect to  $y$ . In other words,  $S'(x) = 0$  is a necessary condition for an interior solution with respect to the optimal steady state. The interpretation of this is the usual that a marginal change in sustainable revenue should equal the best alternative rate of return. With a zero discount rate the best alternative rate of return is zero, and hence the sustainable revenue ought to be maximized.

In the case of multiple solutions with respect to the optimal path, one should choose the one that has appropriate dynamic characteristics. In other words, choose  $y$  such that  $\dot{x} > 0$  for  $x < x^*$  and  $\dot{x} < 0$  for  $x > x^*$ .

An interesting property of the graph of the  $S$  function is that it can be used to characterize the dynamic equilibrium points in the model. The stationary points of



the  $S$ -function,  $S'(x) = 0$ , corresponds to equilibrium points in the phase-space. A local minimum corresponds to a centre, a local maximum corresponds to a saddle point and an inflection point corresponds to a cusp. This property is quite useful as  $S$  is a function in one variable only and therefore can be easily plotted. Hence all equilibrium points in the phase-space can be found by visual inspection of the graph of the  $S$  function no matter how complex the phase-space might be.

### **The feedback rule with discounting**

Equation (4) is in general a highly non-linear ordinary differential equation (ODE). Therefore one can normally not expect to find a closed-form solution. This does not pose a serious problem, however, because when we have the exact solution in the case of zero discounting, it is relatively easy to find very good approximative solutions using perturbation theory, see Nayfeh (1973). Perturbation methods are rather infrequent in economics<sup>2</sup>. The basic idea is to formulate a general problem, find a particular solution and use this as a starting point to find approximate solutions to nearby problems. The difference between the approach used here and the common approach, e.g. Ludwig (1979), is that we use the solution from the case with zero discounting as starting point. As seen from the previous section this is a non-trivial solution that contains most of the complexities and non-linearities in the model, and therefore not many correction terms are needed. The common approach is to use simpler starting points, and therefore more correction terms are needed. The approximation here with only one correction term will, for all practical purposes, be so good that the deviation by far is outweighed by measurement errors in the stock level, etc.

The difference between the "exact" solution and the solution based on perturbation methods for a special case is illustrated in Figure 1. This special case is the same as shown in Figure 2, see "Numerical examples".

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<sup>2</sup>They are, however, frequently used in mathematics and physics.

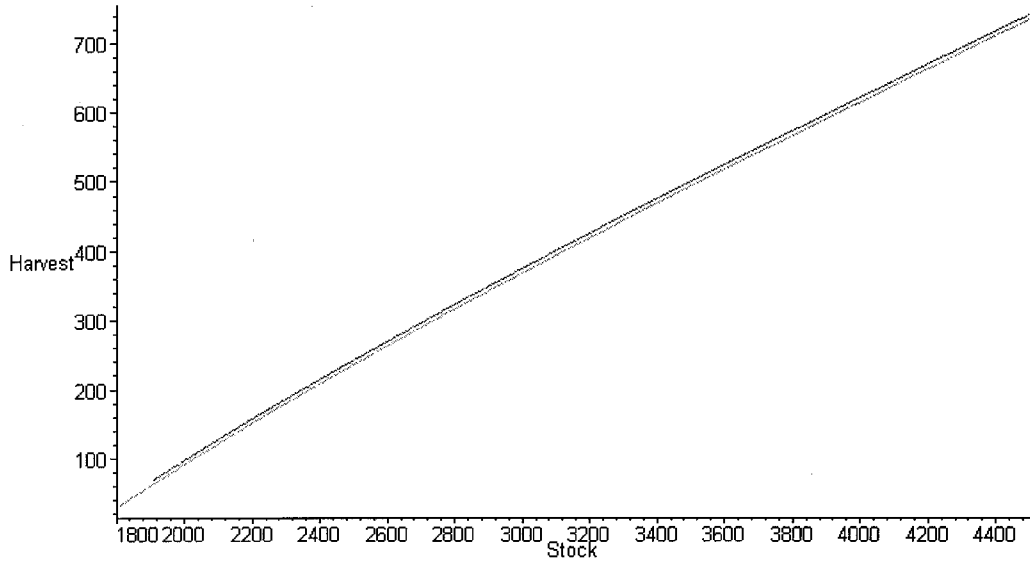


FIG. 1. *Optimal harvest paths with five per cent discounting: perturbation solution and exact solution.*

We use a straightforward perturbation scheme in the control variable and use the discount rate as perturbation parameter. More precisely, the perturbation parameter is the ratio between the discount rate and the maximum (intrinsic) growth rate of the stock. This requires that this ratio is relatively small, otherwise an alternative perturbation scheme must be applied. Usually this ratio is sufficiently small. In fisheries management problems, e.g., only some marine mammals have an intrinsic growth rate that is comparable to any reasonable discount rate.

The perturbation scheme is now given by

$$\begin{aligned}
 y(x) &= y_0(x) + \delta y_1(x) + O(\delta^2), \\
 P(x, y) &= P(x, y_0(x)) + \delta \left[ \frac{\partial}{\partial y} P(x, y_0(x)) \cdot y_1(x) \right] + O(\delta^2) \\
 M(x, y) &= M(x, y_0(x)) + \delta \left[ \frac{\partial}{\partial y} M(x, y_0(x)) \cdot y_1(x) \right] + O(\delta^2),
 \end{aligned}$$

where  $y(x)$  is the complete solution,  $y_0(x)$  is the solution with zero discounting,

$y_1(x)$  is the first correction term, etc. This is then substituted into (4). The scheme represents an asymptotic series as opposed to convergent series. An inherent property of asymptotic series is that the optimal number of correction terms is small, often one or two, whereas convergent series become better the more correction terms are added. In this model it is for all practical purposes sufficient to include only one correction term. The first order correction term to the feedback rule is given by

$$y_1(x) = \frac{\int_{x^*}^x M(z, y_0(z)) dz}{\frac{\partial}{\partial y} P(x, y_0(x))}, \quad P(x, y_0(x)) = P_0,$$

where  $P_0$  is constant and  $x^*$  is the steady state. Notice that  $x = x^*$  does not necessarily imply that  $y_1 = 0$  because both the denominator and the numerator then are zero.<sup>3</sup> Using only one correction term the optimal feedback rule with discounting is then given by

$$y(x) = y_0(x) + \delta y_1(x)$$

which is a fully operational expression for optimal exploitation for any stock level. Again the feedback rule with the appropriate dynamic characteristics should be chosen in the case of multiple solutions; that is,  $\dot{x} > 0$  when  $x$  is below the optimal steady state and vice versa.

The optimal steady state, however, has changed in the case of discounting. This is now given by solving

$$S'(x) = -\delta \cdot \frac{\Pi_y}{f_y} \Big|_{f(x,y)=0} = \delta M(x, y) \Big|_{f(x,y)=0} \quad (6)$$

with respect to  $x$ . Note that  $\frac{\Pi_y}{f_y} \Big|_{f(x,y)=0}$  is a function in  $x$  only. By dividing (6) by  $\delta$  it can be seen that this has the usual interpretation that the present value of all future benefit (left-hand side) should equal the instantaneous benefit (right-hand side). It is readily seen that  $S'(x) = 0$  is a special case of (6) when  $\delta = 0$ .

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<sup>3</sup>From (3)  $P_y = \mathcal{H}_y + \mathcal{H}_M M_y = \mathcal{H}_M M_y = \dot{x} M_y = 0$  in steady state.

## NUMERICAL EXAMPLES

In this section the feedback rule is illustrated by two numerical examples, one based on fisheries management and one based on pollution control. The examples are given to highlight the practical applicability of the model under various circumstances and to show how to use the model in practice.

### Fisheries management

In the first example some stylized facts pertaining to North-East Atlantic cod may be recognized. The net revenue function is given by

$$\Pi = p(y)y - c(x, y)$$

where

$$p(y) = \underline{p} + (\bar{p} - \underline{p})e^{-ay} \quad \text{and} \quad c(x, y) = k\frac{y}{x}.$$

The parameters  $\underline{p}$  and  $\bar{p}$  in the demand function represent minimum price and maximum price respectively. These have been specified to  $\underline{p} = 4$  and  $\bar{p} = 10$ , and the parameter  $a = 0.001$ . The parameter  $k$  in the cost function has been specified to  $k = 10,000$ . The dynamic constraint is given by

$$f(x, y) = g(x) - y$$

where  $g$  may be any function of  $x$ . In this example we use a Gompertz' function of the form

$$g(x) = rx \ln\left(\frac{K}{x}\right)$$

where  $r = 0.25$  and the carrying capacity  $K = 8,000$ . From (2) we have

$$m(x, y) = -\frac{\Pi_y}{f_y} = \underline{p} + (\bar{p} - \underline{p})(1 - ay)e^{-ay} - \frac{k}{x}.$$

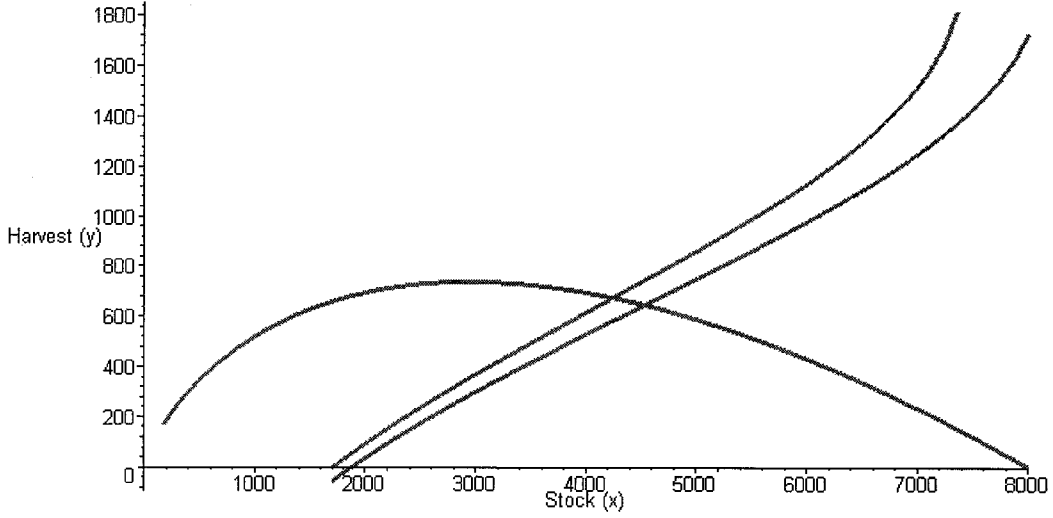


FIG. 2. Optimal harvests as functions of the stock level for zero discounting (lower curve) and five per cent discounting (upper curve) together with the growth function.

Substituting this into the Hamiltonian we get

$$\left[ \underline{p} + (\bar{p} - \underline{p})e^{-ay} \right] y - k \frac{y}{x} + \left( \underline{p} + (\bar{p} - \underline{p})(1 - ay)e^{-ay} - \frac{k}{x} \right) (g(x) - y) = P_0 \quad (7)$$

with zero discounting. Eq. (7) defines the optimal harvest as a feedback control law although it is not possible to solve this equation explicitly. The optimal steady state,  $x^*$ , corresponds to the global maximum of  $S(x)$ , and the constant  $P_0$  is given by  $P_0 = S(x^*) = \max S(x)$ . The optimal harvest,  $y$ , as a function of the stock level,  $x$ , is plotted in Figure 2 together with the  $f$ -function for zero and five per cent discounting.

The optimal steady state with zero discounting is  $x^* = 4529$ , and the corresponding harvest  $y^* = 644$ . This corresponds to  $P_0 = 3184$ .

## Pollution control

The second example is based on pollution control. Assume that pollution,  $a$ , is associated with the production of some good,  $b$ , such that the change in  $a$  is given by

$$\dot{a} = g(b) - d(a) = b^2 - \alpha \cdot \exp(-\beta(a - \gamma)^2).$$

Pollution increases exponentially with production, and the decay of pollution follows a bell-shaped curve. Further, the benefit function is given by

$$B(a, b) = \frac{\ln(b)}{a + 1}.$$

This benefit function has the properties that  $B_a < 0$ ,  $B_b > 0$ ,  $B_{bb} < 0$  and  $B \rightarrow 0$  as  $a \rightarrow \infty$ . In addition to this  $B(0, b) = \ln(b)$ .

From (2) we have

$$m(a, b) = -\frac{B_b}{d_b} = \frac{-1}{2b^2(a + 1)}$$

which inserted into the Hamiltonian yields

$$\frac{\ln(b)}{1 + a} + \frac{-1}{2b^2(a + 1)} [b^2 - \alpha \cdot \exp(-\beta(a - \gamma)^2)] = P_0 \quad (8)$$

in the case with a zero discount rate. In this particular case it is possible to find an explicit closed-form solution for the control variable  $b$  from eq. (8):

$$b(a) = \exp\left(\frac{1}{2}W(\Omega(a)) + \frac{1}{2} + P_0(a + 1)\right) \quad (9)$$

where

$$\Omega(a) = -\alpha \cdot \exp[-\beta(\gamma - a)^2 - 2P_0(a + 1) - 1],$$

and  $W$  denotes the Lambert $W$ -function defined by the expression

$$W(x) \cdot e^{W(x)} = x.$$

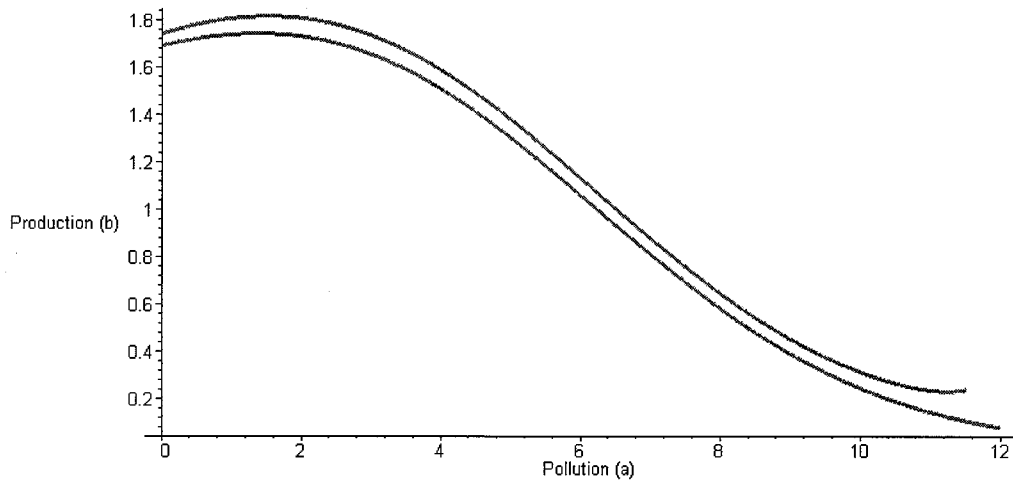


FIG. 3. *Optimal production as functions of the aggregated pollution level for zero discounting (lower curve) and ten per cent discounting (upper curve).*

The numerical specification of the decay function is as follows:  $\alpha = 2.5$ ,  $\beta = 0.08$ ,  $\gamma = 5$ . Then the value of  $P_0$  is found in the usual manner to be 0.084 and the corresponding values of  $a$  and  $b$  are 3.95 and 1.5 respectively. Optimal production as a function of the pollution level, that is equation (9), is illustrated in Figure 3 for zero and ten per cent discounting.

It is seen from this example that it is possible to find explicit feedback laws for the control variable even when the input functions in the model are highly non-linear.

### SUMMARY

In this paper a general dynamic optimization problem has been solved by eliminating the costate variable from the Hamiltonian, and this has been combined with the fact that the Hamiltonian is constant with zero discounting (autonomous case). This yields an exact feedback solution in the case *without* discounting; that is, the exploita-

tion level (control) is given as a function of the resource stock (state variable). In the case *with* discounting the solution has been expanded using perturbation theory in order to find an approximate solution. The approximation is sufficiently good for all practical purposes, and is usually by far outweighed by other sources of uncertainty.

As optimal exploitation of a renewable resource has been found as a feedback rule, this represents an adaptive approach to resource management. The feedback rule is found as an implicit closed-form expression even when the model is completely general. By completely general is meant that the objective function as well as the dynamic constraint are general functions of both the control and the state variable within the class of functions that give a unique solution. It is, of course, assumed that the expressions fulfil the usual regularity requirements. In the general case the feedback rule is given as an implicit expression. It is possible, however, to find explicit closed-form solutions for many special cases. Closed-form solutions, even when they are implicit, make it possible to perform parameter analysis, comparative dynamics, etc., which makes this approach useful for qualitative as well as quantitative studies.

The approach is illustrated by two numerical examples, one based on a fisheries management problem and one based on a pollution control problem. The model can also be applied to non-renewable resources by removing the growth function, and letting the time horizon be determined by the requirement that the shadow value of the resource is zero.

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