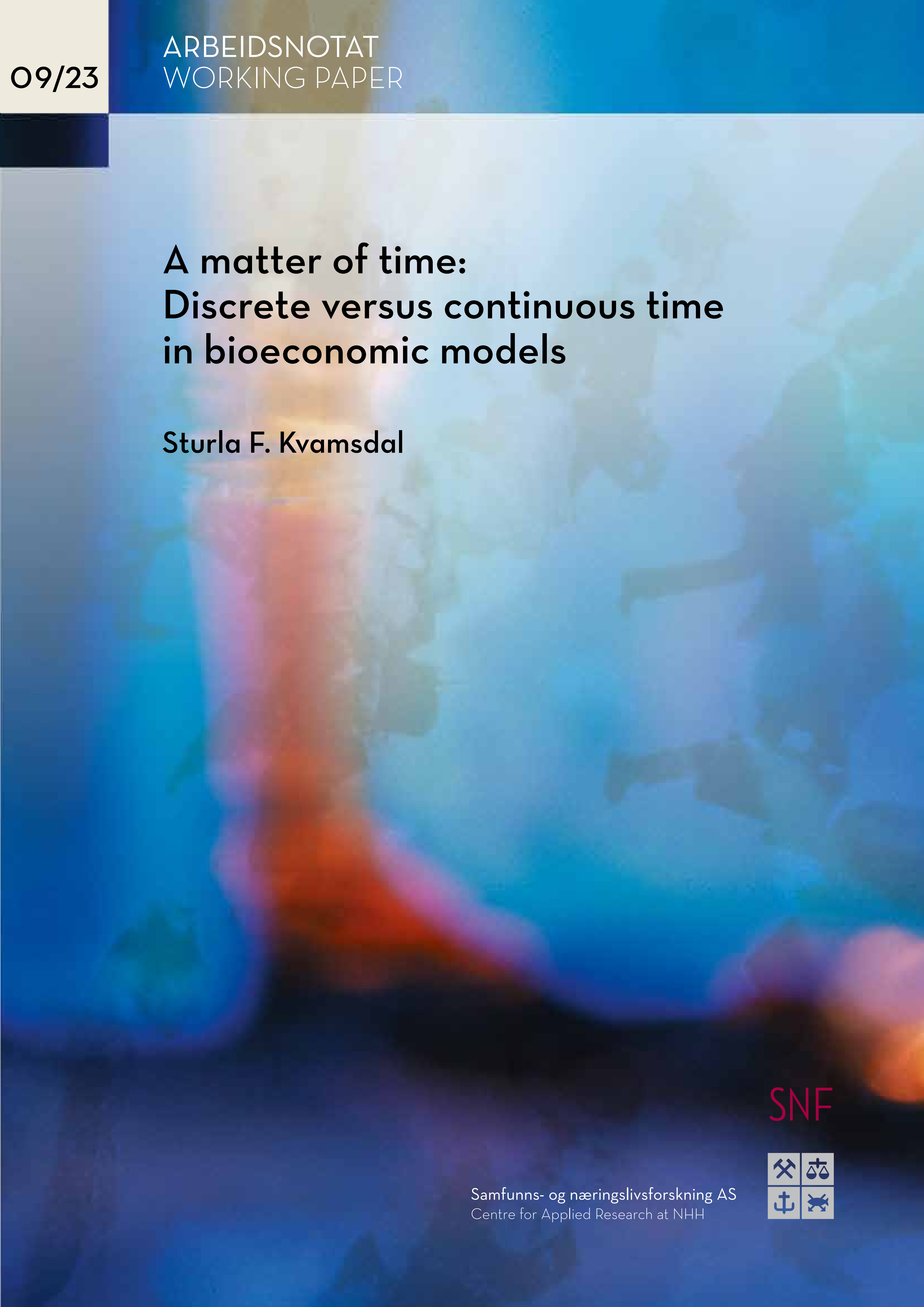

09/23

ARBEIDSNOTAT
WORKING PAPER

A matter of time: Discrete versus continuous time in bioeconomic models

Sturla F. Kvamsdal

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A matter of time: Discrete versus continuous time in bioeconomic models

Sturla F. Kvamsdal

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Abstract

In the context of bioeconomic modeling, I discuss issues related to the use of discrete and continuous time. Using discrete or continuous time for a given modeling problem is not necessarily a matter of preference, but has methodological consequences that should be observed. These include differences in dynamic behavior and the construction of corresponding models for transfer, between models of different type, of functional forms, parameter values, or results. Further, I discuss properties of quota advice based upon a continuous time model.

Keywords: Discrete time modeling, continuous time modeling, surplus production bioeconomic models.

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1 Introduction

A fundamental question in applied modeling is whether to formulate a model in discrete or continuous time. The question has been discussed in various contexts, and has been latent at least since the dawn of modern empirical research. Some economists have shown interest in the question: Medio (1991) discussed problems of chaotic dynamics in relation to business cycle models, and Carlstrom and Fuerst (2003) discussed selected issues and open questions related to discrete and continuous time formulations in macroeconomics more generally. Sims (2008) considered bias and non-intuitive effects of discrete time models in the context of time series analysis. While the general discrepancy between discrete and continuous time models may be reasonably well understood, certain considerations in bioeconomics are worthy of deliberation. Notably, the focus in this essay is on surplus production models, which is where the discussed issues are of relevance. Age-structured models—alternatives to surplus production models in bioeconomic modeling—are fundamentally of a discrete nature although conceptually derived from a continuous framework.

Given the prevalence of models in both discrete and continuous time, the fisheries economist ought to have a natural interest in my topic. But the interest may be culled by a misguided—if not erroneous—belief that using discrete or continuous time is simply a matter of taste or preference. However, discrete and continuous time models have their strengths and weaknesses, and for any given modeling problem, one or the other may be ideal. This idea that the question is a matter of preference may underpin misuse and inconsistencies. For example, adoption of functional forms or parameter values or estimates from other studies into a given model regardless of type (type signifying discrete or continuous time) may lead astray. One cannot generally apply functional forms or parameter values or estimates (the latter follows from the former) in a model of opposite type without further ado, because a given functional form will have different properties, in particular dynamic properties, in the opposite model type. Another example of inconsistency is to compare or incorporate results between models of different types without rigorous transformations.

This matter-of-preference-idea is, admittedly, a subtlety which has no consequence when one works exclusively with one model type. This subtle nature and its lack of consequence in much theoretical work may explain why the idea exists and why inconsistencies go undetected. Nevertheless, the idea is puzzling because I think many bioeconomic researchers know, somewhere deep down, that discrete and continuous models have different properties. A possible explanation could then be that a transformation—the construction of a corresponding model—is believed theoretically possible, and perhaps even that this transformation is trivial. If one has this belief, one may conclude that model type does not really matter and that it is a matter of preference only. But, as I will explain and illustrate in a simple example, construction of a unique corresponding continuous time model for a discrete time model is not generally possible. And the construction of a corresponding discrete time model for a continuous time model may be cumbersome and require some numerical sophistication. Let me hasten to add that some of the mentioned inconsistencies, for example the adoption of discrete time parameter estimates in continuous time models, are present in some of my own early work. As the discussion below shows, dealings with such inconsistencies have led to separate, method-focused research papers. These experiences suggest that this essay has merit.

Another potential explanation for inconsistencies is an implicit assumption that the approximation error is small and nevertheless dominated by other approximations that inherently crop up in bioeconomic modeling (and in most other model-related work; see discussion in Sims, 2008). That such assumptions tend to go unmentioned can mislead readers who may, subconsciously, conclude: Matter of preference. Furthermore, the approximation error may be large.

Much of bioeconomic theory (for example, much of Clark, 1990) relies on continuous time formulations. But in applied work, discrete time models may be more natural and appropriate. Examples include econometric studies and dynamic systems with discrete properties, such as fish stocks with migrating behavior. Discrete time models are also subject to theoretical work, obviously. The construction of corresponding models is thus of primary interest, for example, when theoretical results from a continuous time model are to be compared to—or superimposed upon—an applied, discrete time model. Or when theoretical results are compared between models of different types. However, the literature is surprisingly silent on the question of corresponding models.

Alongside the question of corresponding models, a discussion of some of the different properties of discrete and continuous time models is in place. The discussion shows that properties that are strengths in one setting may be a weakness in the other. Such differences underpin the importance of constructing corresponding models and shows that discrete and continuous time models complement rather than exclude each other.

The aim of this essay is not to derive formal properties or proofs of propositions, but rather to illustrate key differences between model types in simple examples. I do not provide a catalogue of inconsistencies or misuse, and I have no intent to undermine earlier work. To the contrary, I have the greatest respect and gratitude for earlier work because it has helped me develop an understanding of the matter of time in models.

2 Dynamic properties in discrete and continuous time

In discrete time models, time is represented by discrete, usually equidistant, and sequentially ordered points. A state variable may take different values at each point in time. An alternative interpretation, particularly invoked in empirical work, is that model variables are observable only at the discrete times. Either way, the dynamic behavior of variables (how they move) between the discrete times are not specified and is essentially beyond the model. In continuous time models, time is continuous (and sequentially ordered) such that between any two points in time, there exists a third point in time. Model variables are observable at all points in time, and dynamic behavior, how variables move, between points in time is known. Both types of models have strengths and weaknesses, and for a given modeling problem, one or the other may be ideal.

Continuous time is often preferred in theoretical work. The workhorse model *numero uno*, the standard logistic growth law, is:

$$f(x) = rx\left(1 - \frac{x}{k}\right) \tag{1}$$

In (1), x is the stock biomass level, r is called the intrinsic growth rate, and k the carrying capacity. When natural growth is said to follow the standard logistic in continuous time, it usually means that the time derivative is equal to (1), $\mathrm{d}x/\mathrm{d}t = f(x)$, assuming for the moment that no harvest takes place. The first order discrete approximation is:

$$\frac{X_{t+\Delta t} - X_t}{\Delta t} = rX_t\left(1 - \frac{X_t}{k}\right) \tag{2}$$

In (2), X_t is the stock biomass level at time t, and Δt is a constant time step or increment. Equation (2) can be rewritten as follows:

$$X_{t+\Delta t} = X_t + \Delta t \cdot rX_t\left(1 - \frac{X_t}{k}\right) \tag{3}$$

The following equation, clearly inspired by (3), is often used in discrete time models:

$$X_{t+1} = X_t + \mathrm{R}X_t\left(1 - \frac{X_t}{k}\right) \tag{4}$$

The parameter R in (4) is referred to as the growth rate. From (3) and (4), it emerges that $R = \Delta t \cdot r$, that is, $R = r$ requires $\Delta t = 1$. With $\Delta t = 1$, however, equation (2) is not necessarily a good approximation of $\mathrm{d}x/\mathrm{d}t = f(x)$; (2) is only guaranteed to be a good approximation in the limit $\Delta t \to 0$. A common statement is that (2) holds for small Δt. But to estimate a value for R

using equation (4) and subsequently use that value in a continuous time model (1) is not necessarily consistent. In general, care must be taken when using parameter estimates in a continuous time formulation. As pointed out by Sims (2008), discrete time approximations to continuous time models where derivatives appear may be poor, but the discrepancy is often overlooked. Application of Markov Chain Monte Carlo (MCMC) methods makes estimation of differential equation models easier (Sims, 2008). An approach related to MCMC methods is the ensemble Kalman filter, which has been applied to fisheries models (Kvamsdal & Sandal, 2015; Ekerhovd & Kvamsdal, 2017).

Equation (4), the discrete time logistic growth equation, can be derived from the first order approximation of the continuous time logistic growth function, as shown above. This transition brings up the question of corresponding models. To be clear, the discrete time corresponding model of a continuous time model equals the continuous time function at the discrete observation times. Figure 1 illustrates the idea. The left panel shows a curve for how a variable develops in continuous time, and the right panel shows how the discrete time model corresponds to the curve at the discrete observation times. In the simplest cases, such as for the logistic, there exist a closed form discrete time corresponding model. Notably, this closed form is not equation (4), but is the Beverton-Holt growth function (Gyllenberg *et al.* 1997). The differing dynamic behavior of (1) and (4) will be discussed further below. When closed forms are infeasible, numerical methods such as the Runge-Kutta approximation can be applied to discretize a continuous time function (Kvamsdal *et al.* 2017).

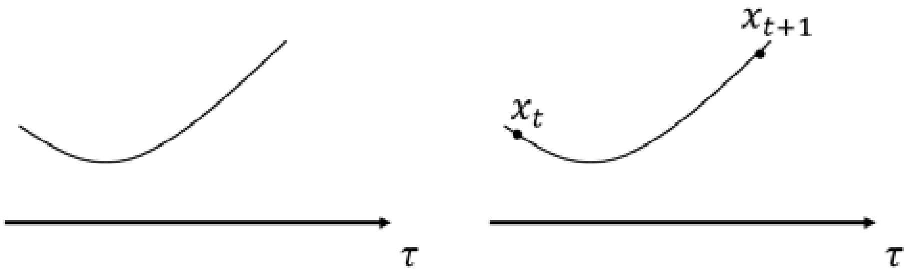

Figure 1: Illustration of a variable developing in continuous time along the time dimension (τ) (left panel), and the discrete time corresponding model (right panel).

The example above illustrates a general point, that for continuous time models, corresponding discrete time models are feasible in theory. Continuous time corresponding models for discrete time models are a different matter, however. For a given discrete time model, a continuous time model that corresponds to the discrete time model at the discrete observation times can be constructed. A spline function, for example, can be specified to pass through the discrete time observations. But this continuous time model is not unique. Figure 2 illustrates the issue. The left panel shows how a discrete model variable develops from one time step to the next. The right panel shows three different continuous time models that correspond to the discrete model at the discrete observation times. The continuous time corresponding model must fill the void between the discrete observation times, and there is no unique way to do this. In a sense, the discrete time function contains less information than a continuous time function, and the continuous time model is underdetermined.

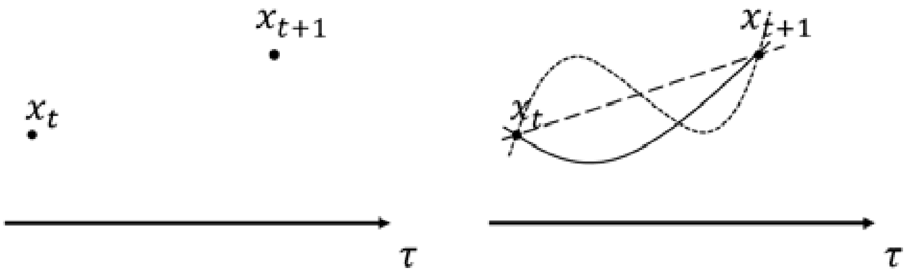

Figure 2: Illustration of a discrete time development from one time step to the next (left panel), and different corresponding continuous time models (right panel).

The dynamic properties of the logistic growth law in continuous and discrete time are fundamentally different. For example, at high stock levels ($X_t \gg k$), equation (4) yields $X_{t+1} < k$. That is, a catastrophe occurs, and from one time step to the next, a very large stock has collapsed. In the extreme, $X_{t+1} < 0$, which obviously is an unrealistic feature. With logistic growth in continuous time, the stock level would rapidly converge to the carrying capacity if $x \gg k$, which is a reasonable phenomenological feature for many fish stocks and is also intuitively appealing. While stock levels in the discrete time formulation that imply $X_{t+1} < 0$ may be beyond the region of interest, any stock level $X_t > k$ imply some kind of collapse if $R > 1$ because then $X_{t+1} < k$ (see derivation below). Further, for any R, there is some stock level $X_t > k$ implying $X_{t+1} < k$. This is a nonintuitive and somewhat disturbing property. Again, the carrying capacity is approached from above for any growth rate in continuous time models.

In general, a hump shaped discrete time growth function, that is, a function with a maximum, can lead to cyclical behavior if the maximum X_t^* is below the carrying capacity. Clark (1990) refers to this feature as overcompensation. If we write the growth function as $X_{t+1} = G(X_t)$, the carrying capacity is where $G(X_t) = X_t$, say X_t^k. Cyclical behavior can emerge if $X_t^* < X_t^k$. If $G(X_t)$ is the logistic and defined by the right-hand side of (4), we can take the derivative and find that the maximum is given by the expression $\frac{k}{2R}(1+R)$. The condition that the maximum is less than the carrying capacity, $\frac{k}{2R}(1+R) < k$, leads to $R > 1$. Cyclical behavior of (4) is discussed in detail by May (1974).

While cyclical behavior may be fascinating in itself, it carries with it features that may be undesirable in a bioeconomic model. A simple illustration may be instructive. The left panel in Figure 3 displays the discrete time logistic growth curve with $R = 2$ and $k = 1$ (red curve). The blue curve is the identity map or the replacement curve, $X_{t+1} = X_t$. The black curve illustrates a time path development with $X_1 = 0.1$ that ends up fluctuating around, and slowly converging upon, the carrying capacity. The right panel in Figure 3 display the same growth curve and the time path with $X_1 = 0.067$, which ends up basically directly at the carrying capacity (with small fluctuations beyond the rounding error). The general point is that the long run behavior varies significantly with the initial stock level. In nature, fluctuations may be more the rule and stability the exception, but such significance of the initial state may be unfortunate for a theoretical model.

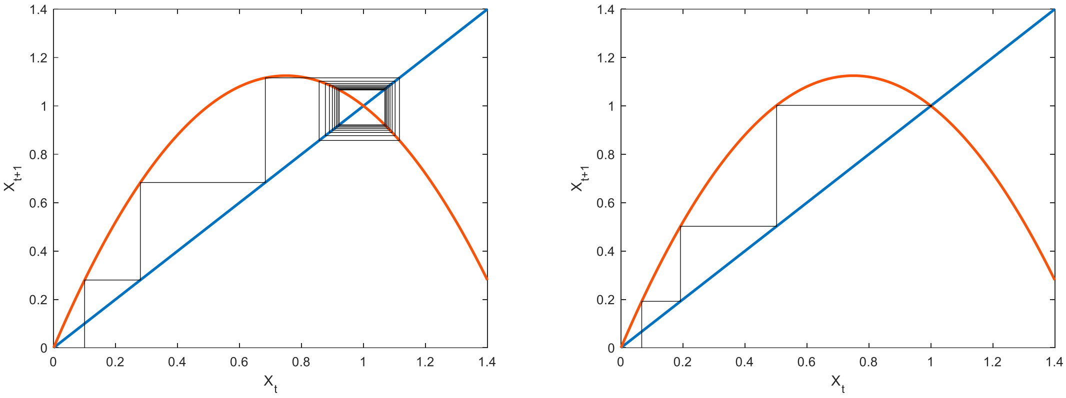

Figure 3: The logistic growth function in discrete time (red curves), with $R = 2$ and $k = 1$, see equation (4), the replacement curve $X_{t+1} = X_t$ (blue curves), and time paths for $X_1 = 0.1$ (left panel) and $X_1 = 0.067$ (right panel).

Local stability of equilibria is formally analyzed in terms of the slope of the growth function. As mentioned, May (1974) provides a comprehensive analysis of the discrete time logistic and shows that for $R > 2$, the equilibrium at the carrying capacity is unstable. Various limit cycles emerge as the growth rate is increased, eventually ending in behavior indistinguishable from chaos. Chaos, appropriately, is difficult to define precisely, but a typical feature is a substantial long-run sensitivity to initial conditions, as vividly captured in the butterfly effect. Propensity for chaos is another dynamic property that is fundamentally different for discrete and continuous time models. As discussed by Lindström (2009), discrete time models have potential for more complicated dynamics. For continuous models, the Poincaré-Bendixson theorem excludes chaotic dynamics for one- and two-dimensional differential equation systems (see Lindström, 2009 and references therein). But as said, in discrete time, chaos can occur in a one-dimensional model. More generally, discrete-time models readily accommodate a rich set of phenomena that require a more extensive structure to be represented in continuous time. Some examples are lags and delays (Clark, 1990, p. 197), hysteresis more generally, catastrophic collapse as discussed above, and seasonal models (Kvamsdal *et al.* 2017, 2020). This richness of discrete-time formulations is difficult to represent in typical continuous-time models because more regularity is imposed upon the latter.

3 Fisheries economics models in discrete and continuous time

Applied in fisheries economics, models formulated in discrete time has the advantage that they typically provide the harvest quota directly. Thus, if the stock-harvest dynamics is described by the following equation:

$$X_{t+1} = \mathrm{F}(\mathrm{X}_\mathrm{t}) - Q_t \tag{5}$$

In (5), $F(X_t)$ is some appropriate, discrete-time growth function, and Q_t is the quota for period t. If time is measured in years, Q_t is the annual quota.

The model in (5) requires consideration of the sequence of events. In (5), growth occurs before harvest takes place. Another way to view this sequencing is in terms of the observation time; the stock level is observed (measured) after each harvesting event. If the sequence of events was the other way around, if harvest takes place before growth, or equivalently, if the stock level is observed after each growth period, the right-hand side of (5) would be formulated as $\hat{F}(X_t - Q_t)$, ($\hat{F}$ may have to differ from F). Obviously, when the stock level is observed does not change the dynamics although a different growth function may be needed, but it affects the problem formulation. That is, the sequence of events impacts how the decision variable Q_t enters the dynamic equation. Generally, sequencing and the implicit assumption that growth occurs without harvest and no growth occurs during harvest make discrete time models somewhat unsuited to describe the dynamics of some fish stocks. Examples include stocks that are exploited throughout the year. Sequencing may also impact how discounting impacts revenues and costs. On the other hand, separation of growth and harvesting periods is a reality in, for example, many salmon fisheries and other fisheries that for example target spawning migrations. Furthermore, discrete time models readily support data from both biological and economic systems that tend to be collected at regular intervals.

In models formulated in continuous time, one typically does not worry about the sequence of events. The canonical stock-harvest dynamics is as follows:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x) - u \tag{6}$$

In (6), $f(x)$ is some continuous-time growth function, and u is the instantaneous harvest rate. That is, in continuous time, growth and harvest happens simultaneously. Continuous time models are usually subject to regularity assumptions about differentiability. Notably, such assumptions are made for analytic convenience and are not necessarily reflections of physical reality. Admittedly, one may find it difficult to argue against a reality that is continuous and somewhat smooth.

The harvest rate u, which is the decision variable, is not the relevant quota, however. The harvest rate is a flow variable; the quota is a stock variable. To derive the associated, say annual, quota, the harvest rate should be integrated over the appropriate time interval. Further, the distribution of harvest in time should follow the harvest rate, something that make continuous time models somewhat unsuited for provisioning of management advice. When fishing quotas are issued, fishers are usually free to fish when they want in a given season. As shown by Huang and Smith (2014), this freedom may be a source of inefficiency in fisheries (see also Smith, 2012 and references therein).

To illustrate the difference between a harvest rate and the resulting, appropriate quota, consider the following management problem:

$$\max_{u \geq 0} \int_0^\infty \mathrm{e}^{-\delta \mathrm{t}} \Pi(x, u) \mathrm{d}t \tag{7}$$

The stock variable x is subject to the dynamic constraint (6); let $f(x)$ be the logistic (1). The discount rate δ is set to 2%. The objective function is given by:

$$\Pi(x, u) = (p - c_1 u) \cdot u - c_2 u / x \tag{8}$$

The objective is nonlinear in u such that the solution is not bang-bang, and there is a stock effect on costs. Notably, stock effects on costs or in the objective function generally are not trivial to accommodate for in discrete time models, and may require auxiliary assumptions (see Clark, 1990). Parameter values for the results below are listed in table A1 in the appendix.

Time is assumed measured in years such that the annual quota $q(x_0)$, where $x(t = 0) = x_0$ denotes the stock level at the start of the year, is given as follows:

$$q(x_0) = \int_0^1 u(x) \mathrm{d}t \tag{9}$$

Figure 4 plots the optimal harvest rate (blue curve) and the associated quota (red curve) as functions of the stock level. Because of the stock effect, the optimal harvest rate is zero for low stock levels (approximately $x < 0.2$). Further, the optimal harvest rate increases with the stock level and the optimal steady state (x^*) is somewhere near $x = 0.75$. To the left of the steady state, the stock will increase over time because the growth is higher than the harvest rate. Thus, for an initial stock level below the steady state, $x_0 < x^*$, the harvest rate will increase over time and the annual quota is higher than the initial harvest rate, $q(x_0) > u(x_0)$. This effect is quite dramatic at low stock levels, and the annual quota is nonzero for stock level as small as $x = 0.05$. (The numerical values provided here are obviously a consequence of the chosen parameter values and are meant only for illustration.) For initial stock levels above the steady state, $x_0 > x^*$, we have the reverse situation such that the annual quota is below the initial harvest rate, $q(x_0) < u(x_0)$.

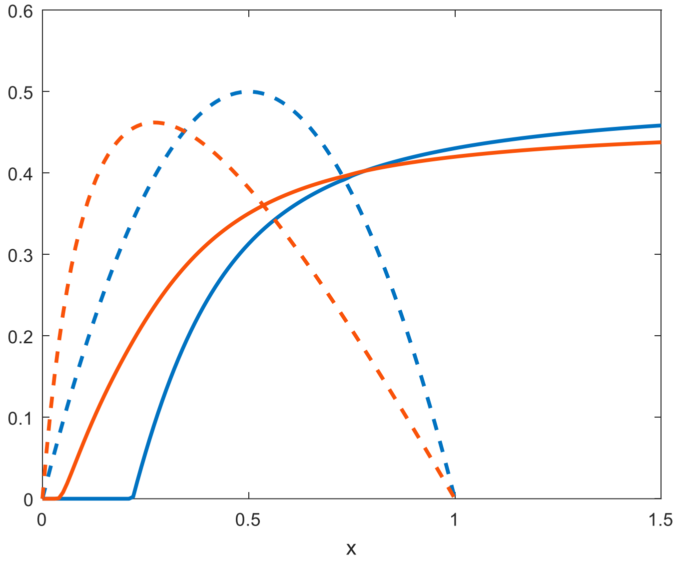

Figure 4: Optimal harvest rate (blue curve), associated annual quota (red curve), logistic growth law (blue dashed curve), and annual growth (red dashed curve).

The discrepancy between the harvest rate and the associated quota can be illustrated also for the stochastic model where the stock dynamics is stochastic. For example:

$$\mathrm{d}x = (f(x) - u)\mathrm{d}t + \sigma x \mathrm{d}B \tag{10}$$

In (10), dB are Brownian increments. Figure A1 in the appendix relates results corresponding to Figure 4.

The results in Figure 4 are generic and may be formulated in a proposition. The proposition concerns stock levels below the equilibrium. A similar proposition could be formulated concerning stock levels above the equilibrium.

Proposition: If, over some given stock level interval (x_0, x_1), the harvest rate is increasing as a function of the stock level, $u'(x) \geq 0$, while below natural growth, $u(x) < f(x)$, the associated quota is higher than the harvest rate, $q(x_0) \geq u(x_0)$.

A formal proof is beyond my ambition in this essay but seems obvious enough.

Figure 4 suggests a seeming paradox because the annual quota intersects the annual growth curve (red dashed curve) at a different stock level than the optimal steady state (where the optimal harvest rate intersects the logistic growth curve). One may be lead to believe that the annual consideration indicates a different steady state than the continuous time model. This conclusion is erroneous because the annual growth curve cannot be compared directly to the quota curve. The annual growth curve relates the integration of the logistic over a time unit, that is, $\int_0^1 f(x)\mathrm{d}t$. The annual growth, in other words, is the growth of the undisturbed (unfished) system.

The annual growth curve is nevertheless of interest because it illustrates differences between discrete time and continuous time considerations. Note that in a discrete time setup, growth is undisturbed, but in a continuous time system, growth and harvest happens simultaneously. The continuous time stock level x can thus be kept, for example, at the optimal steady state indefinitely and provide for annual quotas given by $f(x^*)$. Alternatively, the stock level could be kept at the maximum sustainable yield level and provide higher annual quotas (under a different harvest rate profile, obviously). Figure 4 shows that instantaneous growth $f(x)$ has a higher maximum than the annual (undisturbed) growth, because the annual growth curve integrates over a changing stock level. This is a general fact of discrete time models, that with shorter time steps, where continuous time is the limit, a higher per time production is achievable. Kvamsdal *et al.* (2017) refers to this effect as actualization of the growth rate.

4 Concluding remarks

It matters what type of model one uses in bioeconomic analysis, and the model type should be adapted to the task or research question at hand. Much of what I have highlighted above may be known to the majority of fisheries economists, but my reading of the academic literature has uncovered repeated examples of what one may call bad practice. I am, for example, not aware of any paper that calculates proper annual quotas from a continuous time model, but there are papers that compare an optimal harvest rate with quota levels. I thus find it pertinent to write this essay.

A well-known feature that should alert fisheries economists to the significance of model type, is that in a continuous time model of an open-access fishery, rents are completely dissipated. In a discrete time model, however, inframarginal rents may be positive (see Smith, 2012). The latter is, for example, found by Grainger and Costello (2016) using a discrete time model. Okonkwo and Quaas (2020) find, apparently in contrast, that all fishers have the same outcome in a continuous time model equilibrium, that is, there are no inframarginal rents. There are many different aspects to these studies that may explain the differences, but the different modeling frameworks should alert us that the results may not be directly comparable, or to the possibility that the different results may arise from model type features.

Gyllenberg *et al.* (1997) conclude that population dynamics may be best described by hybrid population models with both continuous and discrete elements. While such an approach may require a more extensive framework, it may help economists avoid some of the pitfalls and inconsistencies discussed above.

There are textbooks that touche upon some of the issues discussed above. To my knowledge, Clark (1990) remains the definitive account. He develops much basic theory in a continuous time framework before deriving several analogous results in a discrete time setup. He also discusses various features, such as delay or seasonal effects, that are more readily effectuated in discrete models. Further topics are cyclical dynamics in discrete models and an analysis in terms of the slope. The question of corresponding models is seemingly not treated, however, and neither is the point that the associated quota differs from a continuous time harvest rate, although he distinguishes these concepts throughout the text. In my humble view, authors of future fisheries economics textbooks would be wise to include these topics.

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Appendix

Table A1: Some parameter values

Parameter	Value	Parameter	Value
r	2	p	1
k	1	c_1	1
σ	0.1	c_2	0.1

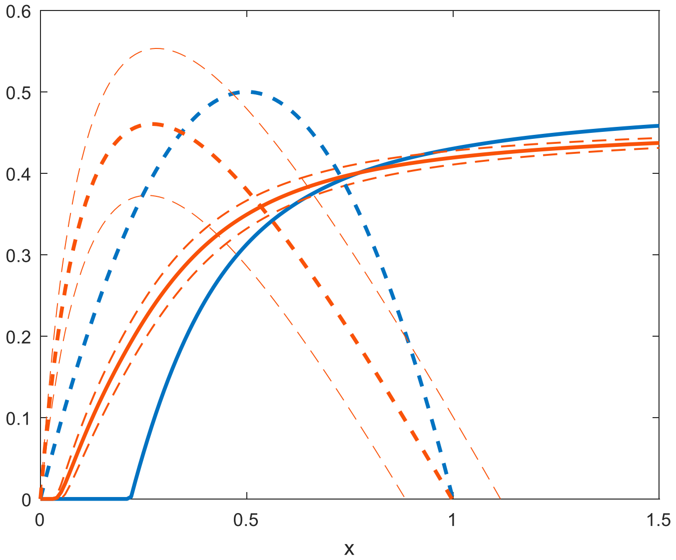

Figure A1: Optimal harvest rate (blue curve), associated annual quota (red curve) plus and minus two standard deviations (thin dashed curves), logistic growth law (blue dashed curve), and annual growth (red dashed curve) plus and minus two standard deviations (thin dashed curves).

In the context of bioeconomic modeling, I discuss issues related to the use of discrete and continuous time. Using discrete or continuous time for a given modeling problem is not necessarily a matter of preference, but has methodological consequences that should be observed. These include differences in dynamic behavior and the construction of corresponding models for transfer, between models of different type, of functional forms, parameter values, or results. Further, I discuss properties of quota advice based upon a continuous time model.

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Trykk: MAKE!Graphics - avd. Bergen