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Equilibrium Prices Supported by Dual Price Functions in Markets with Non-Convexities

by

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Abstract
The issue of finding market clearing prices in markets with non-convexities has had a renewed interest due to the deregulation of the electricity sector. In the day-ahead electricity market, equilibrium prices are calculated based on bids from generators and consumers. In most of the existing markets, several generation technologies are present, some of which have considerable non-convexities, such as capacity limitations and large start up costs. In this paper we present equilibrium prices composed of a commodity price and an uplift charge. The prices are based on the generation of a separating valid inequality that supports the optimal resource allocation. In the case when the sub-problem generated as the integer variables are held fixed to their optimal values possess the integrality property, the generated prices are also supported by non-linear price-functions that are the basis for integer programming duality.

Keywords
Electricity Markets, Pricing, Integer Programming Duality

1. Introduction

A reason for the renewed interest in obtaining market clearing prices in markets with non-convexities is the deregulation of the electricity markets that has taken place recently. In such markets, the non-convexities arise from start-up and shut-down costs of power plants, and minimum output requirements that must be fulfilled in order to run certain plants (see for instance O'Neill et al. (2001, 2002) or Hogan and Ring (2003)). Moreover, non-convexities in a market with uniform linear prices may contribute to and legitimate strategies like "hockey stick" bidding, which may have considerable redistribution effects (Oren (2003)). This may also be regarded as an argument in favor of searching for alternative price mechanisms to deal with the underlying complications that severe non-convexities may create in a market.

As so clearly described by Scarf (1994), “The major problem presented to economic theory in the presence of indivisibilities in production is the impossibility of detecting optimality at the level of the firm, or for the economy as a whole, using the criterion of profitability based on competitive prices.” Scarf illustrates the importance of the existence of competitive prices and their use in the economic evaluation of alternatives by considering an economic equilibrium (Walras). It is assumed that the production possibility set exhibits constant returns to scale, so that there is a profit of zero at the equilibrium prices. Moreover, each consumer evaluates personal income at these prices and market demand functions are obtained by the aggregation of individual utility maximizing demands. Since the system is assumed to be in equilibrium, supply equals demand for each of the goods and services in the economy. A new manufacturing technology, that also possesses constant returns to scale, is to be considered, and, the question is if this new activity is to be used at a positive level or not.

In this setting, the answer is simple and straightforward, as formulated in Scarf (1994): “*If the new activity is profitable at the old equilibrium prices, then there is a way to use the activity at a positive level so that with suitable income redistributions, the welfare of every member of society will increase*.” Furthermore, “*if the new activity makes a negative profit at the old equilibrium prices, then there is no way in which it can be used to improve the utility of all consumers, even allowing the most extraordinary scheme for income redistribution*.” This shows the strength of the pricing test in evaluating alternatives. However, the opportunity of using the pricing test relies on the assumptions made, i.e. that the production possibility set displays constant or decreasing returns to scale. When we on the other hand, have increasing returns to scale the pricing test for optimality might fail, and it is easy to construct examples showing this, for instance by introducing activities with start up costs (a very simple example is provided in O’Neill et al. 2002). With the lack of a pricing test, Scarf introduces as an alternative a quantity test for optimality.

Over time several suggestions have been made to address the problem of finding prices for problems with non-convexities, especially for the case where the non-convexities are modeled using discrete variables. The objective has been to find dual prices and interpretation of dual prices for integer programming problems and mixed integer programming problems. The decisive work of Gomory and Baumol (1963) is addressing this issue, the ideas presented here has later been used to create a duality theory for integer programming problems. Wolsey (1981) gives a good description of this theory, and shows that in the integer programming case we need to expand our view of prices to price-functions in order to achieve interpretable and computable duals (refer also for instance, Alcaly et al. (1966), Scarf (1990), Williams (1996), and Sturmfels (2003)). However, these dual price-functions are rather complex, and other researchers have suggested approximate alternatives, but none of these suggestions have, to our knowledge, been used successfully to analyze equilibrium prices in market with non-convexities.

Recently, O'Neill et al. (2002) presented an innovative method for calculating discriminatory prices, referred to as IP-prices, based on a reformulation of the non-convex optimization problem. The method assumes that the optimal production plan is known and aims at generating market clearing prices, using a non-linear pricing scheme, pricing commodities, start-ups and capacity. Due to the fact that the commodity prices generated by O'Neill et al.'s procedure may possess some unwanted properties, Hogan and Ring (2003) developed a very interesting minimum uplift payment scheme for markets with non-convexities. Their method is similar in spirit to the pricing rules used in the New York and PJM electricity markets. However, the minimum uplift pricing rule is just a simple adjustment of linear pricing to a market in which non-convexities are present, and hence, not in line with integer programming duality theory.

In this paper then, we suggest the use of modified IP-prices as equilibrium prices in markets with non-convexities. The prices are derived using a minor modification of the idea of O'Neill et al. and are based on the generation of a valid inequality that supports the optimal solution and can be viewed as an integer programming version of the separating hyperplane that supports the linear price system in the convex case. The modified IP-prices generated, are based on integer programming duality theory, and it can be shown that there exists a non-linear non-discriminatory price-function, that supports the modified IP-prices.

All three pricing approaches assume that the bidding format is modified in order for the generators to be able to use a multi-part bidding scheme. This is not customary in most of today's existing electricity markets, and the incentive problems that might appear as a result of multi-part bidding protocols, have to be further analyzed.

The paper is organized as follows. In section 2, the non-convex problem at hand is introduced, and the reformulation rendering the IP-prices derived by O'Neill et al. (2002) is presented.

We continue with the partial equilibrium formulation of the problem due to Hogan and Ring (2003), and present their minimum uplift pricing scheme. In section 3, we present the basis for the modified IP-prices and how they can be calculated. In section 4, we compare the three pricing methods on the modified Scarf example used by Hogan and Ring, and also illustrate the corresponding non-linear price functions for a few levels of demand. Finally, some concluding remarks and issues for future research are presented in section 5.

2. An Optimization Model for the Unit Commitment and Dispatch Problem

Hogan and Ring (2003) present an inter-temporal optimization model for economic unit commitment and dispatch. The model allows consideration of both dynamics, i.e. multiple period issues, and transmission constraints. The model is as follows:

$$\text{Max} \sum_{t=1}^{T}\left(\sum_i B_{it}(d_{it}) - \sum_j C_{jt}(g_{jt})\right) - \sum_j S_j z_j$$

subject to

$$L_t(y_t) + \iota^t y_t = 0$$

$$y_t = d_t - g_t \quad \forall\, t$$

$$g_{jt} \geq z_j m_j \quad \forall\, j,t$$

$$g_{jt} \leq z_j M_j \quad \forall\, j,t$$

$$R_{jt}(g_{jt}, g_{jt-1}) \leq 0 \quad \forall\, j,t$$

$$K_t(y_t) \leq 0 \quad \forall\, t$$

$$0 \leq z_j \leq u_j \text{ and integer for all } j$$

where $B_{it}(d_{it})$ denotes a bid-based, well-behaved concave benefit function of demand for customer *i* in time period *t*, $C_{jt}(g_{jt})$ denotes the bid-based, well-behaved convex cost function for output of generator type *j* in time period *t*, and S_j is the bid-based start-up cost for generator type *j*. Note the start-up costs could be time-dependent in a more general version of the model. Also, it is possible to include a constraint that guarantees that the start-up cost is only effective in the period that the generator is started up and hence, the generator can be run without incurring additional start-up cost as long as the generator is active. Shut down cost is also easy to include in the model. Furthermore, m_j and M_j are bid-based minimum and maximum outputs for each generator of type *j* that is committed, z_j is an integer variable showing the number of generators of type *j* committed, u_j is an upper bound on the number of generators of type *j*, and y_t is the vector of net load at each location. For location φ, $y_{\varphi t} = \sum_{i \in \varphi} d_{it} - \sum_{j \in \varphi} g_{jt}$. $L_t(y_t)$ are the losses in period *t* with net demand $\iota^t y_t$ where ι^t =(1,1,….,1). $K_t(y_t)$ represents the transmission constraints for net load y_t in period *t*, and finally, $R_{jt}(g_{jt}, g_{jt-1})$ are ramping limits or other dynamic limits for generator *j*.

The model presented is non-convex due to the integer variables used for modeling the fixed start-up costs and the attached minimum and maximum output constraints. Apart from this, the model is a convex mathematical programming problem, in which we have externalities arising from the transmission constraints.

In this paper we will focus on the non-convexities that are present due to the integer variables. Hence, we will use a single period model without transmission constraints and without losses. Also the model that we use in this paper, has all demand located at a single location. The effects of multiple periods, transmission constraints and losses on the pricing scheme we suggest, will be the focus of future research.

2.1 The Simplified Single-Period Model

In a given period, with demand fixed at *d*, and no losses, ramping or transmission constraints, the problem studied is:

$$\text{Min } \sum_j C_j(g_j) + \sum_j S_j g_j$$

$$\text{subject to} \quad \sum_j g_j = d$$

$$g_j \geq z_j m_j \quad \forall\ j$$

$$g_j \leq z_j M_j \quad \forall\ j$$

$$0 \leq z_j \leq u_j \text{ and integer for all } j$$

$$g_j \geq 0 \quad \forall\ j$$

The simplified model is a least-cost unit commitment and dispatch problem. Our main purpose of studying this simplified model is to present various alternatives for finding equilibrium prices in a market with non-convexities, such as the day-ahead electricity market, in which several different production technologies are present and are competing. The simplified model is of the same type as the allocation model used by Scarf (1994) in his seminal paper on resource allocation and pricing in the presence of non-convexities. Scarf presented, in the absence of usable prices, a quantity test that can be used to verify optimality or as a means to making exchanges that lead to optimality.

Recently, O'Neill et al. (2001) presented an interesting reformulation that generates IP-prices that are interpretable as market clearing prices, and can be viewed as a Walrasian decentralized price-directed equilibrium in the presence of indivisibilities. The reformulation used by O'Neill et al. is accomplished by using knowledge of the optimal solution values for the integer variables, and adding constraints to the model that guarantee that these optimal values are attained. Hence, the procedure and reformulation is not intended to be used in order to calculate the optimal solution, but rather to use the knowledge to obtain interpretable prices that can be used in order to create viable contracts in the market.

2.2 O'Neill et al.'s Reformulation

The single-period problem with the integer commitment variables, z_j, fixed at their known optimal values, z_j^*, is the following:

$$
\begin{aligned}
\text{Min } & \textstyle\sum_j C_j(g_j) + \sum_j S_j z_j \\
\text{subject to } \quad & \textstyle\sum_j g_j = d \\
& g_j \geq z_j m_j \quad \forall\ j \\
& g_j \leq z_j M_j \quad \forall\ j \\
& 0 \leq z_j \leq u_j \\
& g_j \geq 0 \quad \forall\ j \\
& z_j = z_j^* \quad \forall\ j
\end{aligned}
$$

The reformulated problem is a convex programming problem and has an associated dual problem, with dual variables $p, \alpha_j, \beta_j, \gamma_j$ and π_j. O'Neill et al. interpret these dual variables as equilibrium prices, where p is the commodity price, γ_j are the capacity prices, and

α_j, β_j, π_j are discriminatory uplift prices. Based on this, O'Neill et al. determine a set of market clearing contracts, **T**, that can be offered from the auctioneer to the market participants, and that are such that they clear the market. They also show that the IP-prices are optimal solutions to the decentralized optimization problems faced by each generator type, and that in the case where the cost functions take the form $C_j(g_j) = C_j g_j$, i.e. linear cost functions, each generator's optimal profit is exactly equal to zero.

However, as Hogan and Ring (2003) point out, the market clearing prices, or IP-prices, have some properties that are questionable. First, the commodity IP-prices, *p*, can be volatile, i.e. a small change in demand can lead to a large change in the IP-commodity price. Secondly, the IP-uplift charges can be positive or negative and also show a volatile behavior. This will be illustrated in the example in section 4. Hogan and Ring then make use of the reformulation of O'Neill et al. and interpret the IP-equilibrium prices as prices generated from a partial equilibrium model. This is done in order to highlight the importance of the IP-uplift payments that is part of the market settlement system.

2.3 Hogan and Ring's Partial Equilibrium Formulation

Hogan and Ring's partial equilibrium formulation includes the following problems:

A) The problem faced by the system operator, i.e. the market coordinator:

$$\text{Max } p\left(\sum_i d_i - \sum_j g_j\right) - \sum_j \pi_j z_j$$

$$\text{subject to } \quad \sum_j g_j = \sum_i d_i$$

$$0 \le z_j \le u_j \text{ and integer for all } j$$

where *p*, g_j and z_j are variables. It should be pointed out that the system operator only acts as an intermediary market maker that trades electricity with generators and loads and purchases / sells commitment tickets. The assumption is that the system operator is a fully regulated monopolist that through the regulation acts as if it was a price taking profit maximizer.

B) The problem faced by each type of generator:

$$\text{Max } pg_j + \pi_j z_j - C_j(g_j) - S_j z_j$$

$$\text{subject to} \quad g_j \geq z_j m_j$$

$$g_j \leq z_j M_j$$

$$0 \leq z_j \leq u_j \text{ and integer}$$

C) The problem for each consumer:

$$\text{Max } B_i(d_i) + c_i$$

$$\text{subject to} \quad pd_i + c_i \leq \omega_i + \sum_{j>0} s_{ij}\Pi_j$$

with variables p, d_i, and c_i. Here, $s_{ij} \geq 0, \sum_i s_{ij} = 1$ are the share holdings of profits for each consumer and ω_i is the initial wealth endowment for the numeraire good for consumer *i* with consumption level c_i.

Hogan and Ring show that these three problems form a partial equilibrium model, in which the set of prices and quantities $(p, \pi_j, d_j^*, g_j^*, z_j^*)$, constitutes a competitive partial equilibrium. However, the problem faced by the system operator must be reformulated in the same way as is done by O'Neill et al., in order for the set of problems to be well-behaved. I.e. the system operator's problem is:

$$\text{Max } p\left(\sum_i d_i - \sum_j g_j\right) - \sum_j \pi_j z_j$$

$$\text{subject to} \quad \sum_j g_j = \sum_i d_i$$

$$0 \leq z_j \leq u_j \quad \forall\ j$$

$$z_j = z_j^* \quad \forall\ j$$

The IP-prices and quantities support the competitive partial equilibrium. With the equilibrium profit for the system operator $\Pi_0 = p\left(\sum_i d_j^* - \sum_j g_j^*\right) - \sum_j \pi_j z_j^*$, which in the simplified model, with no transmission effects, reduces to the last term. This last term is equal to the net payment to the generators for commitment tickets. From the construct of the revised formulation, including the equality constraints for the integer variables, it is clear that the dual prices π_j can be both positive and negative. This also means that the total profit for the system operator can either be positive or negative. Since the prices are equivalent with the IP-prices in O'Neill et al.'s model, the partial equilibrium interpretation only serves as a guide to how these equilibrium quantities and prices can be interpreted.

As Hogan and Ring make clear, the net payments for the commitment tickets must be collected in form of fixed charges from consumers and/or producers. They are independent of the level of consumption and found in the partial equilibrium model based on the consumers' ownership shares of the system operator's loss or profit. In the day-ahead electricity market, such fixed transfers are not available within the market designs used. However, there are markets in which net payments are collected in the form of an uplift charge, allocated between consumers according to some a priori allocation rule. Because of the volatility of the IP-prices and the fact that the uplift charge can be both positive and negative, Hogan and Ring suggest the use of *minimum* uplift payments and resulting equilibrium prices. The basis for this is that they would like the uplift payments to be nonnegative and the commodity prices less volatile.

2.4 Hogan and Ring's Minimum Uplift Equilibrium Model

Ring (1995) studied approaches for analyzing deviations from the simple perfectly convex equilibrium model. One such deviation of interest is the non-convexities generated from the need to include integer variables in the model formulation. As an approximation, which is close to the ideal situation in the linear case in which a linear price system suffices, Ring suggests an approach in which the necessary total uplift payment is kept as low as possible.

The minimum uplift equilibrium model is based on the idea that given the optimal solution (g^*, z^*), define the nonnegative commitment payments in terms of the uplift requirements needed to make committed and uncommitted generators at least indifferent. This means that for a market clearing price, *p*, the commitment payment to each generator is

$$\pi_j(p) = Max(0, \Pi_j^+ - \Pi_j^*)$$

where $\Pi_j^* = pg_j^* - C_j(g_j^*) - S_j z_j^*$ and Π_j^+ is the optimal value of the problem

$$
\begin{aligned}
&\text{Max } pg_j - C_j(g_j) - S_j z_j \\
&\text{subject to} \quad g_j \geq z_j m_j \\
&\qquad\qquad\quad g_j \leq z_j M_j \\
&\qquad\qquad\quad 0 \leq z_j \leq u_j \text{ and integer} \\
&\qquad\qquad\quad g_j \geq 0
\end{aligned}
$$

The minimum uplift paying rule resulting from this is to pay each generator an amount $\pi_j(p)$ in addition to the payment received for the commodities. The payment $\pi_j(p)$ is conditional on accepting the commitment and dispatch solution.

By construction, the uplift payment is non-negative and the total charge that has to be collected from the customers/consumers is $\sum_j \pi_j$. For each commodity price, there exists a corresponding uplift payment. All these support the competitive market solution. Among all the possible pricing schemes (p, $\pi_j(p)$), the minimum uplift equilibrium pricing scheme is the commodity price for which the total uplift payment is minimal. As is illustrated in the example in section 4, the minimum uplift pricing scheme yields relatively stable commodity prices as a function of demand however, the uplift payments are rather volatile.

3. Modified IP-prices

Along with the minimum uplift pricing of Hogan and Ring and the IP-prices suggested by O'Neill et al., we suggest a third alternative. This alternative is more related to the original uplift prices, but yields more stable commodity prices and uplift charges. The modified IP prices are based on the connection between the IP-prices and the dual information generated when Benders decomposition (Benders (1962)) is used to solve the resource allocation problem.

Benders decomposition is a computational method which is often used to solve models in which a certain set of variables, the complicating variables, are fixed at certain values and the remaining problem and its dual is solved in order to get bounds for the optimal value of the problem and also generate information on how to fix the complicating variables in order to obtain a new solution with a potentially better objective function value. When Benders decomposition is used to solve a nonlinear integer programming problem, as the problem studied in this paper, the problem is partitioned into solving a sequence of “easy” convex optimization problems and their duals, where the complicating integer variables are held fixed. These problems, often called the Benders sub-problems, are generating lower bounds on the optimal objective function value, and yield information that is added to the Benders master problem, a problem involving only the complicating variables. The information generated takes the form of cutting planes.

Here, we are not interested in using Benders decomposition to solve the optimization problem since we assume that we already know the optimal solution. However, viewing the reformulation used by O’Neill et al. as solving a Benders sub-problem in which the complicating integer variables are held fixed to their optimal values reveals useful information concerning the IP-prices generated in the reformulation. The reformulation viewed as a Benders sub-problem is the following:

$$
\begin{aligned}
&\text{Min } \sum_j C_j g_j + \sum_j S_j z_j^* \\
&\text{subject to} \quad \sum_j g_j = d \\
&\qquad g_j \geq z_j^* m_j \quad \forall\ j \\
&\qquad g_j \leq z_j^* M_j \quad \forall\ j \\
&\qquad g_j \geq 0 \quad \forall\ j
\end{aligned}
$$

where z_j^* are the known optimal values for the integer variables. Here we are using the linear version of the problem in order to simplify. However, the results derived are true also for the case when the production function is a general convex function, although in this case nonlinear programming duality has to be used.

The dual to the Benders sub-problem with the integer variables fixed at their optimal values is

$$\text{Max } pd + \sum_j \alpha_j z_j^* m_j - \sum_j \beta_j z_j^* M_j + \sum_j S_j z_j^*$$

$$\text{subject to} \quad p + \alpha_j - \beta_j \leq C_j$$

$$\alpha_j, \beta_j \geq 0$$

The dual of the reformulation used by O'Neill et al. is

$$\text{Max } pd + \sum_j z_j^* \pi_j$$

$$\text{subject to} \quad p + \alpha_j - \beta_j \leq C_j$$

$$-\alpha_j m_j + \beta_j M_j + \pi_j \leq S_j$$

$$\alpha_j, \beta_j \geq 0$$

It is obvious that the optimal dual solution $\left(p^*, \alpha_j^*, \beta_j^*\right)$ yields a feasible solution to the dual of the reformulation and that $\pi_j^* = S_j + \alpha_j^* m_j - \beta_j^* M_j$. This means that the optimal dual variable values of the equality constraints, $z_j = z_j^* \quad \forall\ j$, are in fact the coefficients in the Benders cut that is generated when solving the Benders sub-problem with the integer variables held fixed at their optimal values.

The Benders cut might incorporate coefficients in a valid inequality. The Benders objective function cut has the form

$$p^* d + \sum_j \left(\alpha_j^* m_j - \beta_j^* M_j + S_j\right) z_j$$

From this we can generate an inequality of the form

$$\sum_j \left(\alpha_j^* m_j - \beta_j^* M_j + S_j\right) z_j \geq \sum_j \left(\alpha_j^* m_j - \beta_j^* M_j + S_j\right) z_j^*$$

This inequality is a valid inequality for some problems, whereas for other problems a different set of variables, including both integer-constrained variables and continuous variables, needs to be held fixed in order to generate a supporting valid inequality. From this, it is obvious that the dual information generated in O'Neill et al.'s reformulation and the dual information generated from the Benders sub-problem stated above, is the same.

In linear and convex programming the existence of a linear price vector is based on the use of the Separating Hyperplane Theorem. Based on the convexity assumption, the equilibrium prices are the dual variables or Lagrange multipliers for the market clearing constraints. In the non-convex case it is well known that not every efficient output can be achieved by simple centralized pricing decisions or by linear competitive market prices. If we could find a supporting separating valid inequality that could serve the same purpose as the separating hyperplane in the convex case, we would have a way to construct equilibrium market clearing prices in the non-convex case. However, in order to do so, we need to use a reformulation of the original problem in which the coefficients derived from the Benders sub-problem are coefficients in valid inequality. The problem is that this is not always the case when we fix the *integer* variables to their optimal values.

Based on this observation, it is easy to see why the IP commodity and uplift prices are volatile. The reason for the volatility comes from the fact that for some problems the coefficients in the Benders cut, derived when the integer variables are held fixed to their optimal values, are in fact coefficients from a separating valid inequality for the mixed integer programming problem studied. Hence, the separating supporting valid inequality is the replacement of the supporting hyperplane that is the basis for linear pricing in the convex case. However, for other problems this is not the case. This means that if we are looking at a class of problems for which a supporting valid inequality only includes integer variables, the IP-prices generated will yield a supporting valid inequality in the sense that the inequality supports the optimal solution and is a separating hyperplane. That is, appending this valid inequality to the original problem does not cut off any other feasible solution.

The original Scarf example, with two production technologies, Smokestack and High-Tech, falls into this class of problems. However, Hogan and Ring's modified Scarf problem does not. Even if the problem under study is not within the class of problems in which a supporting valid inequality only includes integer variables, it might be the case that for certain demand values the coefficients for the integer variables generated from the optimal dual variable values are coefficients for these variables in a supporting valid cutting plane, whereas for other demand values they are not. For problems in this class, the IP-commodity price and IP-uplift prices will be volatile.

Hence, in order to derive IP-prices that are supported by a separating valid inequality, it is necessary to regard also some of the continuous variables as complicating variables that are to be held fixed in the reformulated problem. This means that when using the partitioning idea in Benders decomposition in order to calculate the optimal solution, it is clear that only the integer variables are complicating, when “complicating” is interpreted as complicating from a computational point of view. However, when our aim is to generate interpretable prices, another set of variables should be regarded as complicating. We need to find out which variables should be held fixed at their optimal values in order for the Benders cutting plane, derived by solving the Benders sub-problem and it’s dual, to be a valid inequality that supports the optimal solution. When these variables are held fixed at their optimal values, this cutting plane will, when added to the problem, generate commodity prices and uplift charges that have the properties we are looking for, i.e. market clearing and non-volatile. We call the prices derived this way the modified IP-prices. In the example presented below, we will show how these prices can be calculated.

If the problem studied is such that the sub-problem generated when the integer variables are held fixed to their optimal values has the integrality property, it is also possible to derive the corresponding nonlinear price function. The existence of such non-linear price functions and how they can be generated can be found in Wolsey (1981). It should be noted that since we know what we are looking for, i.e. a valid inequality having certain specific coefficients, the generation of the non-linear price function might be easier. Also, we are not searching for a supporting separating inequality that when appended to the original problem yields an integer feasible solution since this is not necessary for our purposes when only interpretable dual information is searched for.

In the general mixed integer case, the approach presented by Wolsey to generate the non-linear price-function compatible with sub-additive integer programming duality, cannot be used. However, if we can generate a valid inequality that supports the mixed integer optimal solution, the prices derived from the original problem with this valid inequality appended, can be interpreted as equilibrium market clearing prices. It is an interesting research question to find out if it is possible to generate a non-linear price function also for this general case.

It should be clear that both the Walrasian interpretation that supports O'Neill et al.'s market clearing contracts and the partial equilibrium model that is used by Hogan and Ring can be reformulated for our modified IP-prices.

4. Hogan and Ring's Adapted Scarf's Example

We use Hogan and Ring's example to illustrate the generation of the modified IP prices that are supported by a nonlinear price function. The example consists of three technologies, Smokestack, High Tech and Med Tech, with the following production costs:

	Smokestack	High Tech	Med Tech
Capacity	16	7	6
Minimum Output	0	0	2
Construction Cost	53	30	0
Marginal Cost	3	2	7
Average Cost	6.3125	6.2857	7
Maximum number	6	5	5

We can formulate Hogan and Ring's allocation problem as a mixed integer programming problem as follows, denoted problem P.

(P) Min $53z_1 + 30z_2 + 0z_3 + 3q_1 + 2q_2 + 7q_3$

subject to

$$q_1 + q_2 + q_3 = D$$

$$16z_1 - q_1 \geq 0$$

$$7z_2 - q_2 \geq 0$$

$$6z_3 - q_3 \geq 0$$

$$-2z_3 + q_3 \geq 0$$

$$-z_1 \geq -6$$
$$-z_2 \geq -5$$
$$-z_3 \geq -5$$

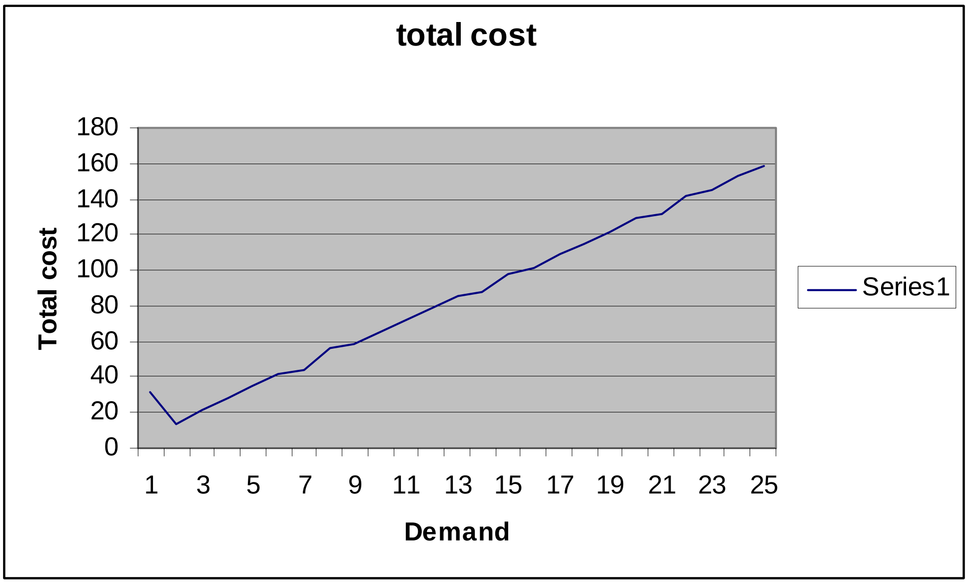
$$q_1, q_2, q_3 \geq 0$$

$$z_1, z_2, z_3 \geq 0 \text{ and integer}$$

Here, D denotes the demand and the construction variables for Smokestack, High Tech and Med Tech are z_1, z_2 and z_3, respectively, whereas q_1, q_2 and q_3 denote the level of production using the corresponding Smokestack, High Tech and Med Tech technologies. Note that for fixed integer values of z_1, z_2, and z_3, the remaining problem in the continuous variables have the integrality property. Hence, the allocation problem is in fact a pure integer programming problem. The reason for this is the special form of the constraint matrix for this example. The optimal solutions for demand levels ranging from 1 to 161 is given in table 1 in the appendix.

The difficulty of finding a linear price structure for this problem is obvious from the graph of the total cost as a function of demand, illustrated in figure 1. The figure only shows the total cost function for demand ranging from 1 to 25. Obviously, as demand grows, the total cost function becomes more linear, and from demand equal to 133, the increase is linear in demand.

Figure 1: Total Cost for Demands Ranging from 1 to 25



In table 2 in the appendix the commodity and uplift prices generated from the three approaches discussed in sections 2 and 3 in this paper, are displayed (IP corresponding to O'Neill et al., Minup to the Hogan and Ring approach, and ModIP is the modified IP prices we suggest in section 3). The three methods presented use three different reformulations to calculate the prices.

O'Neill et al.'s reformulation of problem P is:

$$\text{Min } 53z_1 + 30z_2 + 0z_3 + 3q_1 + 2q_2 + 7q_3$$

$$\text{subject to} \quad q_1 + q_2 + q_3 = D$$

$$16z_1 - q_1 \geq 0$$

$$7z_2 - q_2 \geq 0$$

$$6z_3 - q_3 \geq 0$$

$$-2z_3 + q_3 \geq 0$$

$$-z_1 \geq -6$$
$$-z_2 \geq -5$$
$$-z_3 \geq -5$$

$$z_1 = z_1^*$$

$$z_2 = z_2^*$$

$$z_3 = z_3^*$$

$$q_1, q_2, q_3 \geq 0$$

$$z_1, z_2, z_3 \geq 0 \text{ and integer}$$

where z_1^*, z_2^* and z_3^* represent the optimal integer solution for the specified demand in the range 1 to 161. For some of the demands, this reformulation generates negative uplift prices, which causes the commodity prices and uplift prices to be very volatile as a function of demand. This volatility is illustrated in the two graphs of figures 2 and 3.

Figure 2: Commodity Price, IP

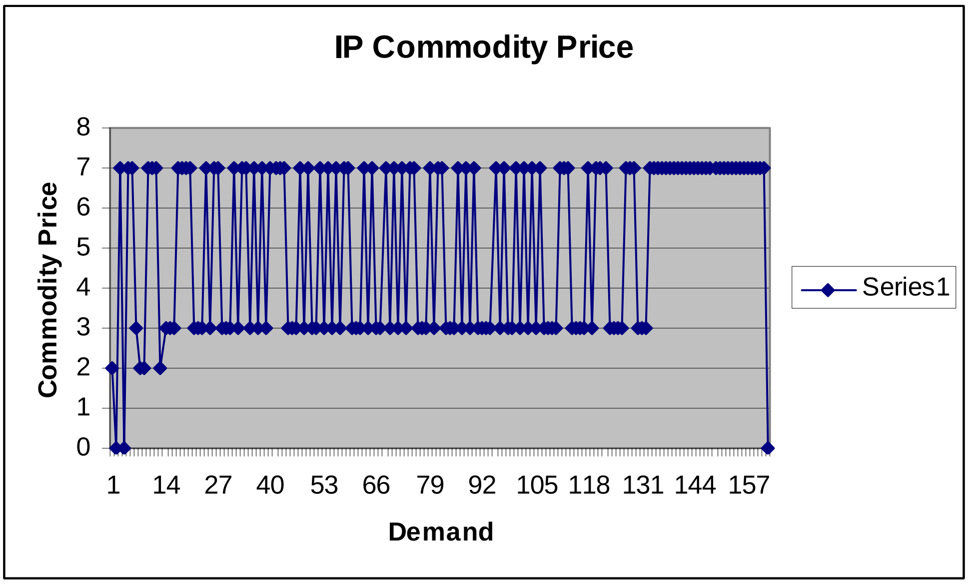
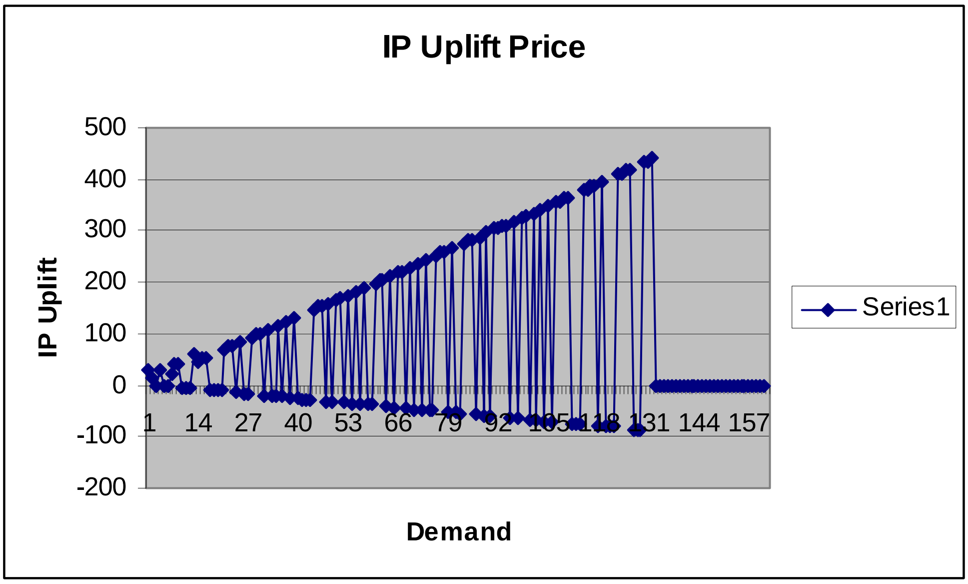


Figure 3: Uplift Price, IP



Hogan and Ring solve the following problem to calculate minimum-uplift commodity and uplift prices: find commodity price, *p*, for which the total uplift payment $\sum_j \pi_j(p)$ is minimal, where the uplift payment to each generator-class is $\pi_j(p) = Max(0, \Pi_j^+ - \Pi_j^*)$, and where $\Pi_j^* = pq_j^* - C_j q_j^* - S_j z_j^*$, the unit cost vector C = (3,2,7), and the start up cost vector S = (53,30,0). The Π_j^+s are the optimal objective function values of the problems:

$$\Pi_j^+ = \text{Max } pq_j - C_j q_j - S_j z_j$$

subject to $m_j z_j \leq q_j \leq M_j z_j$

$z_j \leq u_j$ and integer

From the construction, it is clear that the min uplift commodity prices are less volatile as compared with the IP prices. This is seen in figure 4 for the example. However, the min uplift payments are volatile at a low level, which can be seen from figure 5.

Figure 4: Commodity Price, Minup

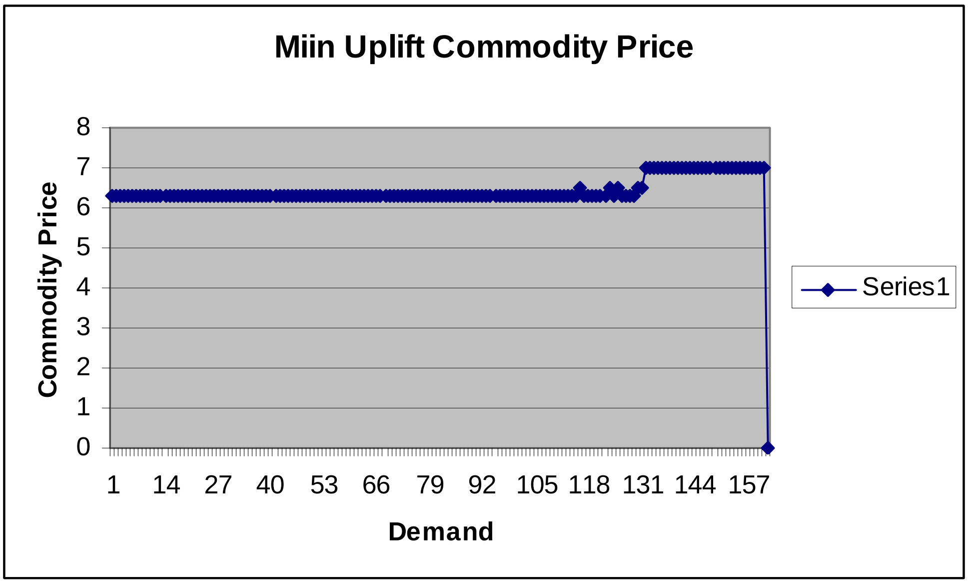
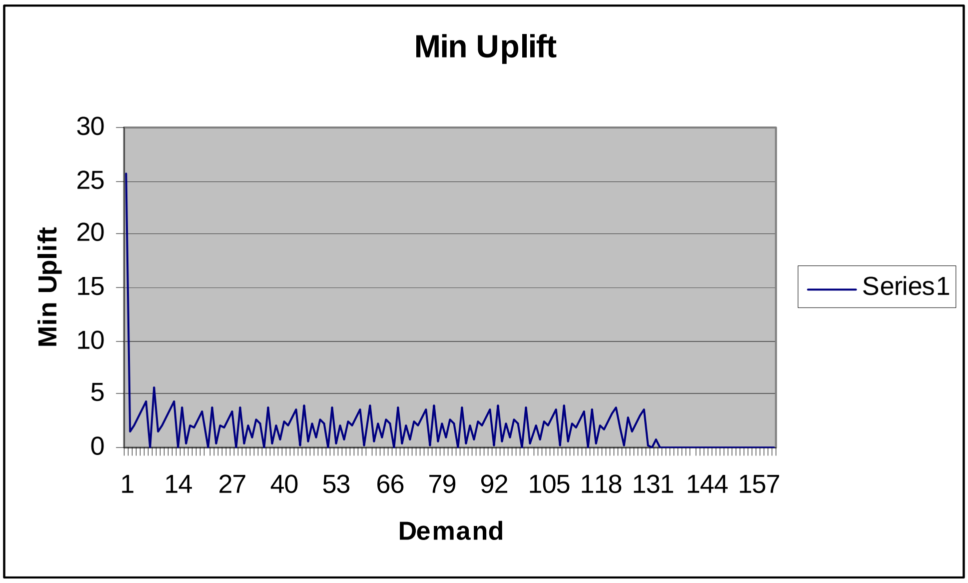


Figure 5: Uplift Price, Minup



For the modified IP prices, the reformulated problem to be solved is:

$$\text{Min}\, 53z_1 + 30z_2 + 0z_3 + 3q_1 + 2q_2 + 7q_3$$

subject to $q_1 + q_2 + q_3 = D$

$$16z_1 - q_1 \geq 0$$

$$7z_2 - q_2 \geq 0$$

$$6z_3 - q_3 \geq 0$$

$$-2z_3 + q_3 \geq 0$$

$$-z_1 \geq -6$$
$$-z_2 \geq -5$$
$$-z_3 \geq -5$$

$$z_1 = z_1^*$$

$$z_2 = z_2^*$$

$$q_3 = q_3^*$$

$$q_1, q_2, q_3 \geq 0$$

$$z_1, z_2, z_3 \geq 0 \text{ and integer}$$

The reason for requiring production of units of the third type to be held fixed ($q_3 = q_3^*$) is that by doing so, the coefficients in the corresponding Benders cut will be coefficients in a valid inequality, a separating hyperplane, that supports the optimal solution for all levels of demand. Hence, by construct the modified IP prices are theoretically consistent with mixed integer programming duality theory.

As seen in the graphs of figures 6 and 7, the modified IP-prices are not volatile, and the modified IP uplift prices form, as expected, an increasing stepwise function of demand up to the point where the problem's continuous relaxation has an integer solution. From that level of demand on, in this example D = 133, linear pricing is sufficient.

Figure 6: Commodity Price, ModIP

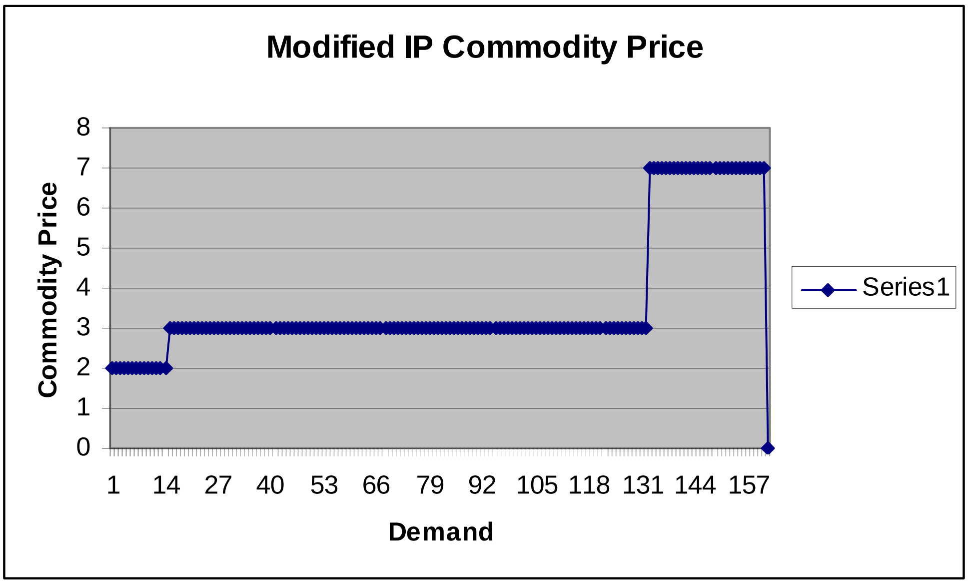
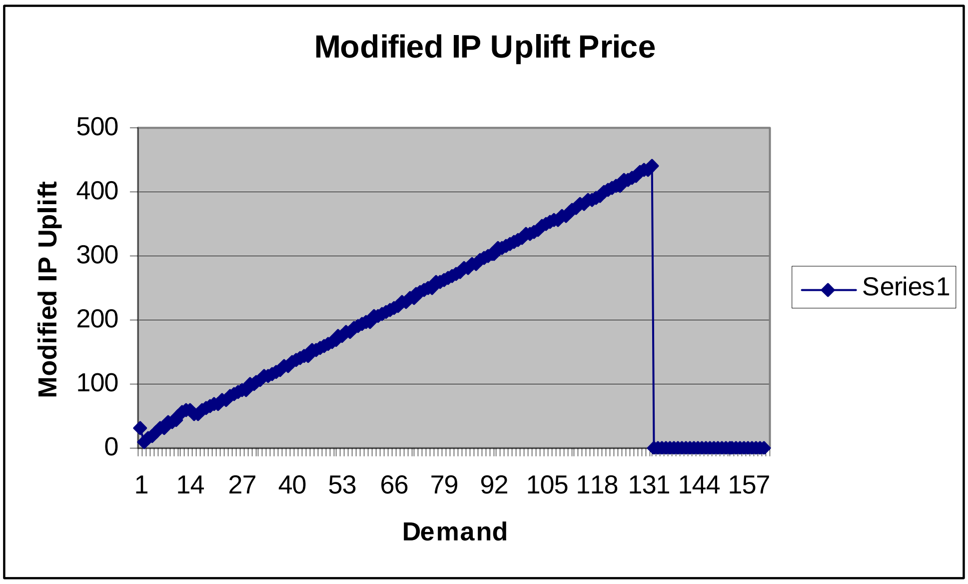


Figure 7: Uplift Price, ModIP



When the modified IP-prices are derived, the following valid inequalities are used: For demand ranging from 1-13, the inequality $53z_1 + 30z_2 + 5q_3 \geq MIP_i$ is a price supporting valid inequality. Here, MIP_i is the modified IP-uplift seen in the table above. For demand ranging from 14-132, the valid inequality that supports the modified IP-prices, is $53z_1 + 23z_2 + 4q_3 \geq MIP_i$, whereas for demands 133-161, the linear programming relaxation of the problem has integer solution, and hence, a linear price is feasible.

In fact, the inequality $53z_1 + 23z_2 + 4q_3 \geq MIP_i$ is modified IP-price supporting for all values of demand in the range 1-132, except for demands equal to 1, 8 and 13, given that the right hand sides for the demand values 2, 3, 4, 5, 6, 7, 9, 10, 11, and 12 are changed to 8, 12, 16, 20, 24, 23, 31, 35, 39, and 43, respectively. However, using this alternative supporting valid inequality to generate the prices, leads to an increased volatility in commodity and uplift prices as can be seen in figures 8 and 9 below, for the demand values between 1 and 20.

Figure 8: Uplift Price, ModIP

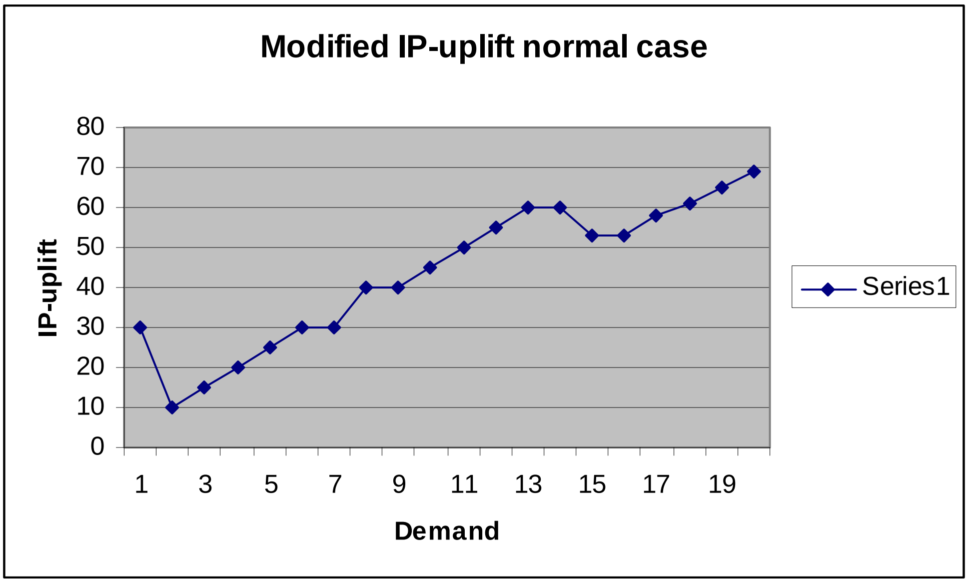
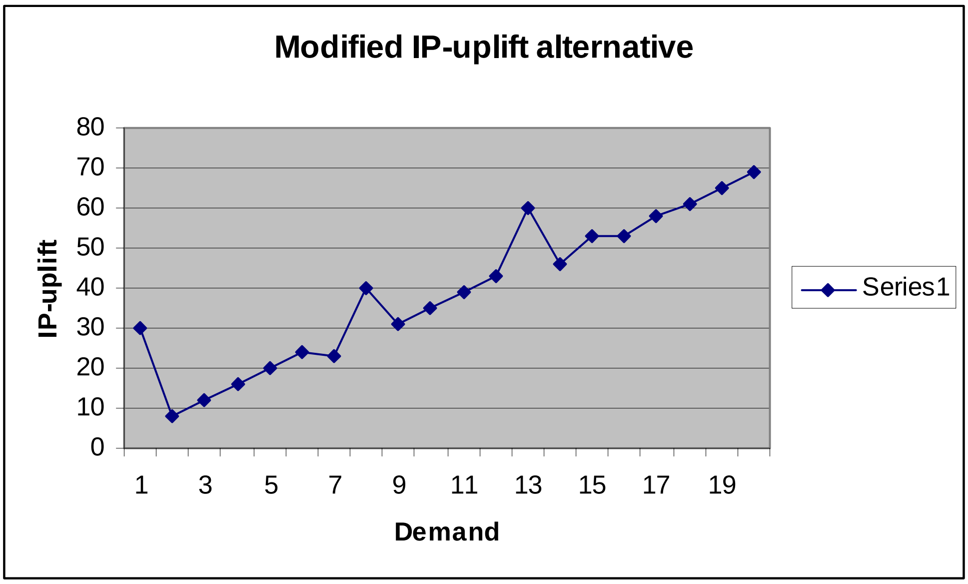


Figure 9: Alternative Uplift Price, ModIP



The modified IP commodity price is stable at the value 2 for demand ranging from 1-13, and then jumps to the value 3 (figure 6). This is when the supporting valid inequality $53z_1 + 30z_2 + 5q_3 \geq MIP_i$ is used for demands 1-13. However, if the alternative is used, the commodity price will be 3 for all demand values ranging from 1-132 except the demands 1, 8 and 13, for which the commodity price is 2.

The modified IP-prices are supported by a non-linear price function and hence, are non-discriminatory. Here, we present the shape of the nonlinear price function for a few demand-values. The non-linear price function for demand equal to 55 can be derived as follows. Adding

$$q_1 + q_2 + q_3 = D$$
$$16z_1 - q_1 \geq 0$$
$$7z_2 - q_2 \geq 0$$

yields the constraint $16z_1 + 7z_2 + q_3 \geq 55$. Adding 7 times the constraint $-z_2 \geq -5$, dividing by 16, and rounding up to the nearest integer value, gives us the constraint $z_1 + q_3 \geq 2$. Taking 3 times the inequality $16z_1 + 7z_2 + q_3 \geq 55$ and adding the inequality $z_1 + q_3 \geq 2$ to this result, we get the next inequality by dividing the result by 7, and rounding up. The resulting inequality is $7z_1 + 3z_2 + q_3 \geq 24$. Finally, the valid inequality $53z_1 + 23z_2 + 4q_3 \geq 182$ is generated by multiplying the inequality $16z_1 + 7z_2 + q_3 \geq 55$ by 9, adding the inequality $z_1 + q_3 \geq 2$ to this intermediate result, and finally, adding 2 times the inequality $7z_1 + 3z_2 + q_3 \geq 24$ to this. The price supporting valid inequality is then derived by dividing by 3 and rounding up to the nearest integer. The nonlinear price function that supports the modified IP prices for the demand level 55 is then given by the following expression:

$$F(*) = 3(1) + 1(3) + \left\lceil \frac{9\big((1)+(2)+(3)\big) + 1\left\lceil \frac{(1)+(2)+(3)+7(7)}{16} \right\rceil + 2\left\lceil \frac{3\big((1)+(2)+(3)\big) + \left\lceil \frac{(1)+(2)+(3)+7(7)}{16} \right\rceil}{7} \right\rceil}{3} \right\rceil$$

Here (*) denotes the coefficient or right hand side of the constraint with the respective number in the formulation of the original mixed integer program.

For demand equal to 56 the nonlinear price function becomes a little bit more complicated. The reason for this is that for this demand level we need to use the technology with the high marginal cost. The optimal uplift for the demand equal to 56 is 187, but the procedure used to construct the price function for demand equal to 55 only generate an uplift of 186 for demand equal to 56. Hence, we have still not generated the price supporting valid inequality. In order to generate the price supporting inequality $53z_1 + 23z_2 + 4q_3 \geq 187$ we need to derive a number of additional valid inequalities. First we generate the two inequalities $30z_1 + 13z_2 + 3q_3 - z_3 \geq 106$ and $30z_1 + 13z_2 + 2q_3 + 3z_3 \geq 106$. These are generated by adding the inequalities $53z_1 + 23z_2 + 4q_3 \geq 186$ and $7z_1 + 3z_2 + q_3 \geq 25$, and to the result add one of the inequalities $-2z_3 + q_3 \geq 0$, $6z_3 - q_3 \geq 0$, dividing by two and rounding up. Also the inequality $14z_1 + 6z_2 + q_3 \geq 49$ is easily generated by adding the inequalities $53z_1 + 23z_2 + 4q_3 \geq 186$ and $16z_1 + 7z_2 + q_3 \geq 56$, dividing the result by five and rounding up.

Another inequality needed to generate the price supporting valid inequality is the inequality $37z_1 + 16z_2 + 3q_3 + z_3 \geq 132$. This inequality comes from adding three of the previously generated inequalities namely, $30z_1 + 13z_2 + 3q_3 - z_3 \geq 106$, $30z_1 + 13z_2 + 2q_3 + 3z_3 \geq 106$ and $14z_1 + 6z_2 + q_3 \geq 49$, dividing the result by two and rounding up. The next inequality that needs to be generated, is the inequality which stems from 2 times the inequality $37z_1 + 16z_2 + 3q_3 + z_3 \geq 132$ and 3 times the inequality $14z_1 + 6z_2 + q_3 \geq 49$, added together with the inequality $-2z_3 + q_3 \geq 0$. Using division by two and rounding up, this yields the inequality $58z_1 + 25z_2 + 5q_3 \geq 206$. Finally, the price supporting inequality is generated by taking three times the inequality $16z_1 + 7z_2 + q_3 \geq 56$, adding to the previously generated inequality $58z_1 + 25z_2 + 5q_3 \geq 206$, and dividing by two.

In order to illustrate the non-linear price function supporting the demand levels 1-13, we choose to generate the non-linear function supporting the equilibrium for demand equal to 1. The following three inequalities are easily derived, $z_1 + z_2 + z_3 \geq 1$, $z_1 + z_2 + q_3 \geq 1$, and $q_1 + q_2 + z_3 \geq 1$, using the fact that $q_1 + q_2 + q_3 = 1$, $16z_1 - q_1 \geq 0$, $7z_2 - q_2 \geq 0$, and $6z_3 - q_3 \geq 0$, adding, dividing, and rounding up. With these three inequalities added, the optimization problem has the objective function value 32 and the solution is integral. The dual variables can be used to generate the inequality $30z_1 + 30z_2 + 20q_1 + 20q_2 + 25q_3 \geq 50$, subtracting 20 times the equality constraint yields $30z_1 + 30z_2 + 5q_3 \geq 30$, which by adding 23 times the inequality $z_1 \geq 0$, generates the desired valid inequality.

It should be noted that we do not suggest that the non-linear price function supporting the optimal solution and the modified IP prices should be generated in practical applications. However, the prices generated using this idea, are due to the existence of a non-linear non-discriminatory price function, and theoretically sound.

5. Conclusions and Issues for Future Research

In this paper we have shown that the modified IP-prices yield commodity and uplift charges that have some nice properties. The use of the optimal solution to generate a reformulation that generates a supporting valid inequality to the resource allocation optimization problem is also presented. This means that the modified IP-prices, for problems which can be regarded as pure integer programming problems, are consistent with the basis in integer programming duality. Knowing the coefficients of a supporting valid inequality means that we know that, in this case, there exists a non-discriminatory non-linear price-function that together with the affine capacity and product prices yield equilibrium prices in markets with non-convexities. Also, knowing the coefficients of the supporting valid inequality makes it easier to construct the non-linear price-function as compared with the case where these coefficients are unknown. Hence, the modified IP-prices have a non-linear non-discriminatory price-function supporting them, and although they seem to be discriminatory, they can also be viewed as a compact representation of the supporting non-linear price-function, and hence, are in fact non-discriminatory. For the more general mixed integer programming case, the modified IP-prices are supported by a separating valid inequality. However, the existence of a non-linear price function and how it can be generated is in this case more complicated and would be an interesting question for further research.

How the knowledge of the existence of a separating supporting valid inequality that can be used to derive equilibrium prices in markets with non-convexities should be used to support a bidding format and a contract mechanism that are incentive compatible, remains another interesting question for future research. Other research questions that should be addressed, are how elastic demand will effect and can be dealt with in markets with non-convexities, if and how the uplift charges should be collected among the customers, and how this cost allocation should influence the design of market clearing contracts. If it is satisfactory just to have a theoretical foundation for the use of modified IP-prices, or if procedures that actually generates a corresponding non-discriminatory price-function needs to be further investigated

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Appendix

Table 1: Optimal Solutions

Demand	Smoke Stack Number	Smoke Stack Output	High Tech Number	High Tech Output	Med Tech Number	Med Tech Output	Total Cost
1	0	0	1	1	0	0	32
2	0	0	0	0	1	2	14
3	0	0	0	0	1	3	21
4	0	0	0	0	1	4	28
5	0	0	0	0	1	5	35
6	0	0	0	0	1	6	42
7	0	0	1	7	0	0	44
8	0	0	1	6	1	2	56
9	0	0	1	7	1	2	58
10	0	0	1	7	1	3	65
11	0	0	1	7	1	4	72
12	0	0	1	7	1	5	79
13	0	0	2	13	0	0	86
14	0	0	2	14	0	0	88
15	1	15	0	0	0	0	98
16	1	16	0	0	0	0	101
17	0	0	2	14	1	3	109
18	1	16	0	0	1	2	115
19	1	16	0	0	1	3	122
20	1	16	0	0	1	4	129
21	0	0	3	21	0	0	132
22	1	15	1	7	0	0	142
23	1	16	1	7	0	0	145
24	0	0	3	21	1	3	153
25	1	16	1	7	1	2	159
26	1	16	1	7	1	3	166
27	1	16	1	7	1	4	173
28	0	0	4	28	0	0	176
29	1	15	2	14	0	0	186
30	1	16	2	14	0	0	189
31	0	0	4	28	1	3	197
32	2	32	0	0	0	0	202
33	1	16	2	14	1	3	210
34	2	32	0	0	1	2	216
35	0	0	5	35	0	0	220
36	2	32	0	0	1	4	230
37	1	16	3	21	0	0	233
38	0	0	5	35	1	3	241
39	2	32	1	7	0	0	246
40	1	16	3	21	1	3	254
41	2	32	1	7	1	2	260
42	2	32	1	7	1	3	267
43	2	32	1	7	1	4	274
44	1	16	4	28	0	0	277
45	2	31	2	14	0	0	287
46	2	32	2	14	0	0	290
47	1	16	4	28	1	3	298
48	3	48	0	0	0	0	303
49	2	32	2	14	1	3	311
50	3	48	0	0	1	2	317
51	1	16	5	35	0	0	321
52	3	48	0	0	1	4	331

53	2	32	3	21	0	0	334
54	1	16	5	35	1	3	342
55	3	48	1	7	0	0	347
56	2	32	3	21	1	3	355
57	3	48	1	7	1	2	361
58	3	48	1	7	1	3	368
59	3	48	1	7	1	4	375
60	2	32	4	28	0	0	378
61	3	47	2	14	0	0	388
62	3	48	2	14	0	0	391
63	2	32	4	28	1	3	399
64	4	64	0	0	0	0	404
65	3	38	2	14	1	3	412
66	4	64	0	0	1	2	418
67	2	32	5	35	0	0	422
68	4	64	0	0	1	4	432
69	3	48	3	21	0	0	435
70	2	32	5	35	1	3	443
71	4	64	1	7	0	0	448
72	3	48	3	21	1	3	456
73	4	64	1	7	1	2	462
74	4	64	1	7	1	3	469
75	4	64	1	7	1	4	476
76	3	48	4	28	0	0	479
77	4	63	2	14	0	0	489
78	4	64	2	14	0	0	492
79	3	48	4	28	1	3	500
80	5	80	0	0	0	0	505
81	4	64	2	14	1	3	513
82	5	80	0	0	1	2	519
83	3	48	5	35	0	0	523
84	4	63	3	21	0	0	533
85	4	64	3	21	0	0	536
86	3	48	5	35	1	3	544
87	5	80	1	7	0	0	549
88	4	64	3	21	1	3	557
89	5	80	1	7	1	2	563
90	5	80	1	7	1	3	570
91	4	63	4	28	0	0	577
92	4	64	4	28	0	0	580
93	5	79	2	14	0	0	590
94	5	80	2	14	0	0	593
95	4	64	4	28	1	3	601
96	6	96	0	0	0	0	606
97	5	80	2	14	1	3	614
98	6	96	0	0	1	2	620
99	4	64	5	35	0	0	624
100	6	96	0	0	1	4	634
101	5	80	3	21	0	0	637
102	4	64	5	35	1	3	645
103	6	96	1	7	0	0	650
104	5	80	3	21	1	3	658
105	6	96	1	7	1	2	664
106	6	96	1	7	1	3	671
107	5	79	4	28	0	0	678
108	5	80	4	28	0	0	681
109	6	95	2	14	0	0	691
110	6	96	2	14	0	0	694
111	5	80	4	28	1	3	702
112	6	96	2	14	1	2	708

113	6	96	2	14	1	3	715
114	5	79	5	35	0	0	722
115	5	80	5	35	0	0	725
116	6	95	3	21	0	0	735
117	6	96	3	21	0	0	738
118	5	80	5	35	1	3	746
119	6	96	3	21	1	2	752
120	6	96	3	21	1	3	759
121	6	96	3	21	1	4	766
122	6	96	3	21	1	5	773
123	6	95	4	28	0	0	779
124	6	96	4	28	0	0	782
125	6	95	4	28	1	2	793
126	6	96	4	28	1	2	796
127	6	96	4	28	1	3	803
128	6	96	4	28	1	4	810
129	6	96	4	28	1	5	817
130	6	95	5	35	0	0	823
131	6	96	5	35	0	0	826
132	6	95	5	35	1	2	837
133	6	96	5	35	1	2	840
134	6	96	5	35	1	3	847
135	6	96	5	35	1	4	854
136	6	96	5	35	1	5	861
137	6	96	5	35	1	6	868
138	6	96	5	35	2	7	875
139	6	96	5	35	2	8	882
140	6	96	5	35	2	9	889
141	6	96	5	35	2	10	896
142	6	96	5	35	2	11	903
143	6	96	5	35	2	12	910
144	6	96	5	35	3	13	917
145	6	96	5	35	3	14	924
146	6	96	5	35	3	15	931
147	6	96	5	35	3	16	938
148	6	96	5	35	3	17	945
149	6	96	5	35	3	18	952
150	6	96	5	35	4	19	959
151	6	96	5	35	4	20	966
152	6	96	5	35	4	21	973
153	6	96	5	35	4	22	980
154	6	96	5	35	4	23	987
155	6	96	5	35	4	24	994
156	6	96	5	35	5	25	1001
157	6	96	5	35	5	26	1008
158	6	96	5	35	5	27	1015
159	6	96	5	35	5	28	1022
160	6	96	5	35	5	29	1029
161	6	96	5	35	5	39	1036

Table 2: Commodity and Uplift Charges

Demand	Commodity prices			Uplift charges		
	IP	Minup	ModIP	IP	Minup	ModIP
1	2	6.286	2	30	25.714	30
2	0	6.286	2	14	1.429	10
3	7	6.286	2	0	2.143	15
4	0	6.286	2	28	2.857	20
5	7	6.286	2	0	3.571	25
6	7	6.286	2	0	4.286	30
7	3	6.286	2	23	0	30
8	2	6.286	2	40	5.714	40
9	2	6.286	2	40	1.429	40
10	7	6.286	2	-5	2.143	45
11	7	6.286	2	-5	2.857	50
12	7	6.286	2	-5	3.571	55
13	2	6.286	2	60	4.286	60
14	3	6.286	2	46	0	60
15	3	6.286	3	53	3.714	53
16	3	6.286	3	53	0.429	53
17	7	6.286	3	-10	2.143	58
18	7	6.286	3	-11	1.857	61
19	7	6.286	3	-11	2.571	65
20	7	6.286	3	-11	3.286	69
21	3	6.286	3	69	0	69
22	3	6.286	3	76	3.714	76
23	3	6.286	3	76	0.429	76
24	7	6.286	3	-15	2.143	81
25	3	6.286	3	84	1.857	84
26	7	6.286	3	-16	2.571	88
27	7	6.286	3	-16	3.286	92
28	3	6.286	3	92	0	92
29	3	6.286	3	99	3.714	99
30	3	6.286	3	99	0.429	99
31	7	6.286	3	-20	2.143	104
32	3	6.286	3	106	0.857	106
33	7	6.286	3	-21	2.571	111
34	7	6.286	3	-22	2.286	114
35	3	6.286	3	115	0	115
36	7	6.312	3	-22	3.688	118
37	3	6.312	3	122	0.375	122
38	7	6.312	3	-25	2.063	127
39	3	6.312	3	129	0.75	129
40	7	6.312	3	-26	2.438	134
41	7	6.312	3	-27	2.125	137
42	7	6.312	3	-27	2.813	141
43	7	6.312	3	-27	3.5	145
44	3	6.312	3	145	0.188	145
45	3	6.312	3	152	3.875	152
46	3	6.312	3	152	0.563	152
47	7	6.312	3	-31	2.25	157
48	3	6.312	3	159	0.938	159
49	7	6.312	3	-32	2.625	164
50	3	6.312	3	167	2.313	167
51	3	6.312	3	168	0	168
52	7	6.312	3	-33	3.688	171
53	3	6.312	3	175	0.375	175
54	7	6.312	3	-36	2.063	180
55	3	6.312	3	182	0.75	182
56	7	6.312	3	-37	2.438	187
57	3	6.312	3	190	2.125	190

58	7	6.312	3	-38	2.813	194
59	7	6.312	3	-38	3.5	198
60	3	6.312	3	198	0.188	198
61	3	6.312	3	205	3.875	205
62	3	6.312	3	205	0.563	205
63	7	6.312	3	-42	2.25	210
64	3	6.312	3	212	0.938	212
65	7	6.312	3	-43	2.625	217
66	3	6.312	3	220	2.313	220
67	3	6.312	3	221	0	221
68	7	6.312	3	-44	3.688	228
69	3	6.312	3	228	0.375	228
70	7	6.312	3	-47	2.063	233
71	3	6.312	3	235	0.75	235
72	7	6.312	3	-48	2.438	240
73	3	6.312	3	243	2.125	243
74	7	6.312	3	-49	2.813	247
75	7	6.312	3	-49	3.5	251
76	3	6.312	3	251	0.188	251
77	3	6.312	3	258	3.875	258
78	3	6.312	3	258	0.563	258
79	7	6.312	3	-53	2.25	263
80	3	6.312	3	265	0.938	265
81	7	6.312	3	-54	2.625	270
82	7	6.312	3	-55	2.313	273
83	3	6.312	3	274	0	274
84	3	6.312	3	281	3.688	281
85	3	6.312	3	281	0.375	281
86	7	6.312	3	-58	2.063	286
87	3	6.312	3	288	0.750	288
88	7	6.312	3	-59	2.438	293
89	3	6.312	3	296	2.125	296
90	7	6.312	3	-60	2.813	300
91	3	6.312	3	304	3.5	304
92	3	6.312	3	304	0.188	304
93	3	6.312	3	311	3.875	311
94	3	6.312	3	311	0.563	311
95	7	6.312	3	-64	2.25	316
96	3	6.312	3	318	0.938	318
97	7	6.312	3	-65	2.625	323
98	3	6.312	3	326	2.313	326
99	3	6.312	3	327	0	327
100	7	6.312	3	-66	3.688	334
101	3	6.312	3	334	0.375	334
102	7	6.312	3	-69	2.063	339
103	3	6.312	3	341	0.750	341
104	7	6.312	3	-70	2.438	344
105	3	6.312	3	349	2.125	349
106	7	6.312	3	-71	2.813	353
107	3	6.313	3	357	3.5	357
108	3	6.313	3	357	0.188	357
109	3	6.313	3	364	3.875	364
110	3	6.313	3	364	0.563	364
111	7	6.313	3	-75	2.25	369
112	7	6.313	3	-76	1.938	372
113	7	6.313	3	-76	2.625	376
114	3	6.313	3	380	3.313	380
115	3	6.313	3	380	0	380
116	3	6.52	3	387	3.48	387
117	3	6.313	3	387	0.375	387

118	7	6.313	3	-80	2.063	392
119	3	6.313	3	395	1.75	395
120	7	6.313	3	-81	2.438	399
121	7	6.313	3	-81	3.125	403
122	7	6.313	3	-81	3.813	407
123	3	6.520	3	410	1.844	410
124	3	6.313	3	410	0.188	410
125	3	6.520	3	418	2.805	418
126	3	6.312	3	418	1.563	418
127	7	6.312	3	-86	2.25	422
128	7	6.312	3	-86	2.938	426
129	7	6.312	3	-86	3.625	430
130	3	6.520	3	433	0.207	433
131	3	6.520	3	433	0	433
132	3	7	3	441	0.688	441
133	7	7	7	0	0	0
134	7	7	7	0	0	0
135	7	7	7	0	0	0
136	7	7	7	0	0	0
137	7	7	7	0	0	0
138	7	7	7	0	0	0
139	7	7	7	0	0	0
140	7	7	7	0	0	0
141	7	7	7	0	0	0
142	7	7	7	0	0	0
143	7	7	7	0	0	0
144	7	7	7	0	0	0
145	7	7	7	0	0	0
146	7	7	7	0	0	0
147	7	7	7	0	0	0
148	7	7	7	0	0	0
149	7	7	7	0	0	0
150	7	7	7	0	0	0
151	7	7	7	0	0	0
152	7	7	7	0	0	0
153	7	7	7	0	0	0
154	7	7	7	0	0	0
155	7	7	7	0	0	0
156	7	7	7	0	0	0
157	7	7	7	0	0	0
158	7	7	7	0	0	0
159	7	7	7	0	0	0
160	7	7	7	0	0	0
161	7	7	7	0	0	0