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prudence: a context-free interpretation**

by

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The values of relative risk aversion and prudence: a context-free interpretation*

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Abstract In this paper we apply to multiplicative lotteries the idea of preference for "harm disaggregation" that was used for additive lotteries in order to interpret the signs of successive derivatives of a utility function. In this way, we can explain in general terms why the values of the coefficients of relative risk aversion and relative prudence are usually compared respectively to 1 and 2. We also show how these values partition the sets of risk averse and/or prudent decision makers into two subgroups.

Keywords: Relative Risk Aversion, Relative Prudence

JEL Classification: D81

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1 Introduction

Recently, Eeckhoudt - Schlesinger (2006) have given an interpretation of the signs of successive derivatives of a von Neumann-Morgenstern (vNM) utility function ($u(\cdot)$) by reference to a preference for harm disaggregation applied to additive lotteries. Of course, the sign of successive derivatives of $u(\cdot)$ already gives much information about the decision maker's preferences since it indicates the direction of various attitudes towards risks. However it is silent about the intensity of such preferences which—for the second and third derivatives—is usually characterized by the value of the coefficients of relative risk aversion (RRA) and relative prudence (RP).

As far as RRA is concerned, the importance of its level (and of its behavior with respect to wealth, here denoted x) has been known for a long time. Indeed, in a series of papers, dealing with portfolio or savings decisions as well as contingent claims models (such as Hahn (1970), Rothschild - Stiglitz (1971), Fishburn - Porter (1976), Mitchell (1994), Chiu and Madden (2007)), it appears that many comparative statics results often depend, among other things, upon a comparison between unity and the value of the RRA coefficient (defined as $-x \frac{u''(x)}{u'(x)}$). This literature has been recently surveyed by Meyer - Meyer (2005). Notice also that in the debate around the equity premium puzzle, the value taken by the (constant) coefficient of relative risk aversion plays a central role.

Since the concept of prudence is more recent than that of risk aversion, the notion of relative prudence (defined as $-x \frac{u'''(x)}{u''(x)}$) is much less discussed than that of relative risk aversion. However, the scant literature that exists

suggests that the benchmark value for RP is 2 (see, e.g., Hadar - Seo (1990) and Choi - Kim - Snow (2001)). In these papers, it appears indeed that a second order dominant shift in the return of a risky asset increases its demand if RP is lower than 2. These results are summarized in Gollier (2001, pp 60-61).

As the models discussed in these papers suggest, the comparisons between RRA and unity on the one hand and between RP and 2 on the other, are indications of the intensity of risk aversion or prudence, respectively. In this sense, the benchmark values of 1 and 2 for risk aversion and prudence partition the sets of risk averse and prudent decision makers each into two subgroups: those who are a little risk averse or prudent and those who are a very risk averse or prudent.

So far, the literature has discussed these benchmark values in a specific institutional context, to wit that of competitive markets for risky assets.¹ The purpose of this paper is to present a comparison of simple lotteries which enables us to elicit in general terms whether relative risk aversion (relative prudence) exceeds the values of 1 (2). The advantage of our approach is twofold. First, it does not rely upon a specific institutional context, and second, it is easily amenable to an experimental implementation. Our procedure applies to multiplicative lotteries an approach similar to the one adopted by Eeckhoudt - Schlesinger (2006) for additive lotteries. The transition from the additive case to the multiplicative one will enable us to interpret in simple

¹There are two exceptions that we are aware of. The first is a paper by Choi - Menezes (1985) who discuss the value of RRA from the concept of a "probability premium". More recently, White (2008) underlines the pertinence of the comparison between RP and 2 in a bargaining game framework.

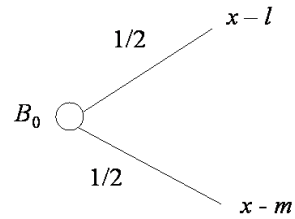
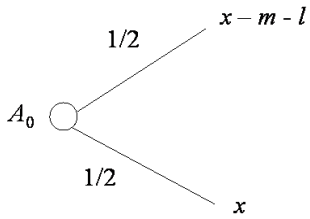
and general terms the benchmark values for *RRA* and *RP*.

Our paper is organized as follows. In the next section, after a brief exposition of the preference for harm disaggregation applied to additive lotteries, we illustrate the equivalent concept for multiplicative risks. The comparison between the two cases gives the intuition for the results presented in section 3 where the benchmark values are formally obtained. We then briefly conclude.

2 Multiplicative risks and the preference for harm disaggregation

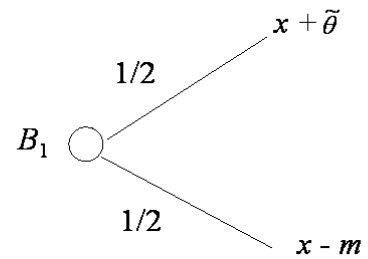
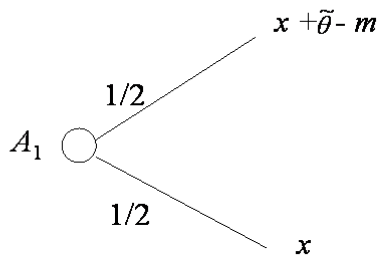
In order to introduce the case of multiplicative risks, we briefly present the additive lotteries used in Eeckhoudt - Schlesinger (2006) to interpret the signs of the second and third derivatives of $u(\cdot)$.

Consider a decision maker endowed with an initial wealth, x , who faces the prospect of two losses ($-l$ and $-m$) occurring each with probability $\frac{1}{2}$. If this individual exhibits a preference for harm disaggregation, he will prefer B_0 to A_0 where



With these additive losses it is easily shown that $B_0 \succeq A_0$ implies and is implied by risk aversion ($u''(\cdot) < 0$).

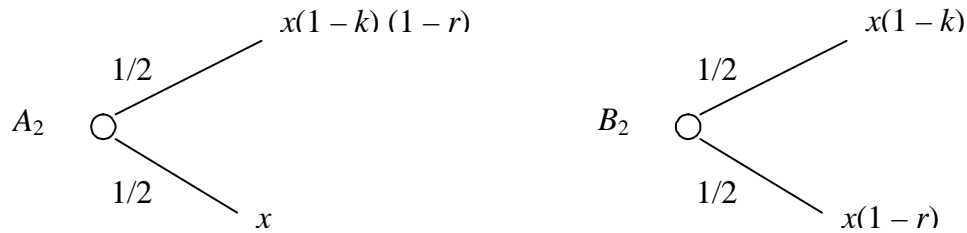
In order to define prudence replace one of the two losses (say $-l$) by a zero mean risk $\tilde{\theta}$ which is also a harm for a risk averse decision maker. Preference for harm disaggregation then means a preference for B_1 to A_1 where



Again the additive nature of the harms leads to the conclusion that $B_1 \succeq A_1$ implies and is implied by prudence ($u''' > 0$).

Turning now to the multiplicative counterpart, the two potential losses

are expressed as shares of wealth denoted respectively k and r with $0 < k < 1$ and $0 < r < 1$. The decision maker can now apportion the harms in two different ways yielding lotteries A_2 and B_2 defined by



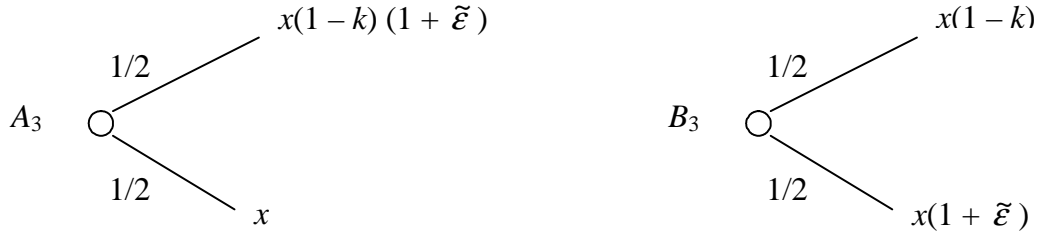
The preference relationship between B_2 and A_2 is now more subtle than it was for the choice between B_0 and A_0 . As for B_0 , the disaggregation of harms that occurs in B_2 gives an advantage to this lottery. However A_2 has also its own advantage: the proportional loss $(1-r)$ now applies to a lower wealth level ($x(1-k)$) so that the expected final wealth of A_2 exceeds that of B_2 . Hence it is clear that risk aversion alone (linked to the preference for harm disaggregation) cannot justify a preference for B_2 . As we formally show in the next section, for B_2 to be preferred to A_2 , risk aversion must be strong enough in the sense that $RRA \geq 1$. Another and equivalent way of interpreting the comparison between B_2 and A_2 (and also below for the comparison between B_3 and A_3) is to notice that B_2 contains "inner" results while A_2 has "extreme" ones. Indeed we necessarily have

$$x(1-k)(1-r) < x(1-k) \text{ and } x(1-r) < x.$$

Of course this feature of B_2 is attractive for risk averse decision makers. However, as we observed before, the expected value of lottery A_2 is higher and thus RRA must exceed one for B_2 to be chosen.

To elicit the intensity of prudence, replace one of the proportional losses

(say $-r$) by a zero mean risky return, $\tilde{\varepsilon} \in [-1, \infty)$, which is disliked by a risk averse decision maker. Hence we now have to compare lotteries A_3 and B_3 defined as:



The harms are better apportioned in B_3 than in A_3 since the two harms never jointly occur in each of the two states of nature and this disaggregation of the two harms is positively appreciated by a prudent individual. But A_3 has—for a risk averse individual—the advantage of a lower variance² since the random return is applied to a lower wealth ($x(1-k)$ instead of x for B_3). Hence, a decision maker will prefer B_3 only if prudence is high enough. We show in the next section that this occurs when $RP \geq 2$.

3 Benchmark values

We now examine how the comparison between the lotteries described in section 2 are expressed in an expected utility (EU) framework and we consider the case of a risk averse ($u''(\cdot) < 0$) and prudent ($u'''(\cdot) > 0$) decision maker.

In proposition 1 we establish a one to one link between the choice among B_2 and A_2 on the one hand and the benchmark value for RRA on the other

²Notice that -contrary to what happened for A_2 and B_2 - the lotteries A_3 and B_3 have the same expected value of final wealth.

hand. Proposition 2 does the same for the choice among B_3 and A_3 and the benchmark value for RP .

Proposition 1 *For an EU risk averse decision maker, $B_2 \succeq A_2$ for any pair $(k, r) \in (0, 1)^2$ if and only if $RRA(X) \geq 1$ for any wealth $X > 0$.*

Proof. Consider a decision maker, endowed with initial wealth x , faces with the two lotteries A_2 and B_2 . Then we have that for any pair $(k, r) \in (0, 1)^2$,

$$\begin{aligned}
& B_2 \succeq A_2 \\
& \iff \\
& \frac{1}{2}u[x(1-k)] + \frac{1}{2}u[x(1-r)] \geq \frac{1}{2}u[x(1-k)(1-r)] + \frac{1}{2}u[x] \\
& \iff \\
& u[x(1-k)] - u[x] \geq u[x(1-k)(1-r)] - u[x(1-r)]. \quad (1)
\end{aligned}$$

(i) *sufficiency.* Define a function $v(\cdot)$ such that $v(r, k; x) \stackrel{\text{def}}{=} u[x(1-k)(1-r)] - u[x(1-r)]$. Then $B_2 \succeq A_2$ iff $v(0, k; x) \geq v(r, k; x)$ for all $k \in (0, 1)$, $x \in R_0^+$. A sufficient condition for $B_2 \succeq A_2$ is thus that

$$\begin{aligned}
& v_r(r, k, x) \leq 0 \\
& \iff \\
& -x(1-k)u'[x(1-k)(1-r)] + xu'[x(1-r)] \leq 0 \\
& \iff \\
& u'[x(1-r)] \leq (1-k)u'[x(1-k)(1-r)].
\end{aligned}$$

Now define function w such that $w(r, k, x) \stackrel{\text{def}}{=} (1 - k) u' [x(1 - k)(1 - r)]$. A sufficient condition for $B_2 \succeq A_2$ (all $r \in (0, 1)$, $x \in R_0^+$), is then that $w(r, k, x)$ is an increasing function in k , that is

$$\begin{aligned}
w_k(r, k, x) &\geq 0 \quad (\text{all } (k, r) \in (0, 1)^2) \\
&\Updownarrow \\
&-u' [x(1 - k)(1 - r)] \\
-x(1 - r)(1 - k) u'' [x(1 - k)(1 - r)] &\geq 0 \quad (\text{all } (k, r) \in (0, 1)^2) \\
&\Updownarrow \\
1 + X \frac{u'' [X]}{u' [X]} &\leq 0 \quad (\text{all } X > 0).
\end{aligned}$$

So that, $RRA \geq 1$ (all $X > 0$) implies $B_2 \succeq A_2$.

(ii) *necessity*. Consider the gambles B_2 and A_2 with $r = k = \delta$, a small positive number. Using a second order Taylor expansion of $u[x(1 - \delta)]$ and $u[x(1 - \delta)^2]$ around $\delta = 0$, inequality (1) can be written as

$$\begin{aligned}
-u''(x)x^2\delta^2 &\geq u'(x)x\delta^2 + O(\delta^3) \\
&\Updownarrow \\
-\frac{u''(x)x}{u'(x)} &\geq 1 + \frac{1}{u'(x)x} \frac{O(\delta^3)}{\delta^2},
\end{aligned}$$

with $\lim_{\delta \rightarrow 0} \frac{O(\delta^3)}{\delta^2} = 0$. We have established that $B_2 \succeq A_2$ for any pair $(k, r) \in (0, 1)^2$ implies $RRA(X) \geq 1$ for any $X > 0$. ■

At this stage, notice the difference between additive and multiplicative risks. For additive risks, as shown in Eeckhoudt - Schlesinger (2006), concav-

ity of the utility function (i.e., risk aversion) is sufficient to justify a preference for disaggregating additive sure harms. Our analysis shows that for multiplicative harms, matters are less simple. In this case—as already mentioned in section 2—harm disaggregation conflicts with a reduction in the mean final wealth and the first effect dominates if and only if RRA exceeds 1. In the contrary case, the mean wealth effect dominates and A_2 is preferred to B_2 .³

Another interpretation of this result can be obtained by considering the logarithmic utility function ($u(x) = \ln(x)$) for which RRA is constant and equal to unity.⁴ In this case, $A_2 \sim B_2$ since

$$\frac{1}{2} \ln(x(1-k)) + \frac{1}{2} \ln(x(1-r)) = \frac{1}{2} \ln(x(1-k)(1-r)) + \frac{1}{2} \ln(x).$$

If one concavifies $\ln(x)$ by taking an increasing and concave transformation $h(\cdot)$ ($h' > 0$ and $h'' < 0$), the resulting utility function $v(x) = h[\ln(x)]$ will exhibit a coefficient of RRA which exceeds unity (see appendix) so that B_2 is then preferred to A_2 . In fact, with the logarithmic utility the apportionment effect and the mean wealth one exactly compensate each other. By concavifying the logarithmic utility (so that $RRA > 1$) more weight is given to the apportionment effect and B_2 is then strictly preferred to A_2 .

We now turn, in proposition 2, to the relationship between relative prudence and the choice among lotteries B_3 and A_3 .

³For example, Chiu and Madden (2007) obtain that some criminal activities are less desirable when risk increases if the individual admits a RRA smaller than 1.

⁴We thank a referee for pointing out this interpretation. Notice that in an important paper on multiplicative background risks, Franke - Schlesinger - Stapleton (2006) use also this tool.

Proposition 2 For an EU risk averse and prudent decision maker, $B_3 \succeq A_3$ for any $k \in (0, 1)$ and $\tilde{\varepsilon} \in [-1, \infty)$ if and only if $RP(X) \geq 2$ for any wealth $X > 0$.

Proof. Consider the possibility of a zero-mean risk of return, $\tilde{\varepsilon}$. Lottery B_3 is preferred to lottery A_3 iff for all zero-mean random variables, $\tilde{\varepsilon} \in [-1, \infty)$, and all k in $(0, 1)$,

$$\begin{aligned} \frac{1}{2}u[x(1-k)] + \frac{1}{2}Eu[x(1+\tilde{\varepsilon})] &\geq \frac{1}{2}Eu[x(1-k)(1+\tilde{\varepsilon})] + \frac{1}{2}u[x] \\ \Downarrow \\ Eu[x(1+\tilde{\varepsilon})] - u[x] &\geq Eu[x(1-k)(1+\tilde{\varepsilon})] - u[x(1-k)]. \end{aligned} \quad (2)$$

(i) *sufficiency.* As previously, define function $v(\cdot)$ such that $v(k, x) \stackrel{\text{def}}{=} Eu[x(1-k)(1+\tilde{\varepsilon})] - u[x(1-k)]$. Then $B_3 \succeq A_3$ (for all $k \in (0, 1)$) iff $v(0, x) \geq v(k, x)$ (all $k \in (0, 1), x \in R_0^+$). A sufficient condition for $B_3 \succeq A_3$ is then that

$$\begin{aligned} v_k(k, x) &\leq 0 \\ \Downarrow \\ -xE(1+\tilde{\varepsilon})u'[x(1-k)(1+\tilde{\varepsilon})] + xu'[x(1-k)] &\leq 0 \\ \Downarrow \\ u'[x(1-k)] &\leq E(1+\tilde{\varepsilon})u'[x(1-k)(1+\tilde{\varepsilon})]. \end{aligned}$$

We now define function $w(\cdot)$ such that $w(k, \varepsilon, x) \stackrel{\text{def}}{=} (1+\varepsilon)u'[x(1-k)(1+\varepsilon)]$. Remembering that $E\tilde{\varepsilon} = 0$ and $var\tilde{\varepsilon} > 0$, we can write that $B_3 \succeq A_3$ (for

all $k \in (0, 1)$, all $x \in R_0^+$) if $Ew(k, \tilde{\varepsilon}, x) \geq w(k, E\tilde{\varepsilon}, x)$. This condition is satisfied if $w(\cdot)$ is strictly convex for all $\varepsilon \in [-1, \infty)$, that is

$$\begin{aligned}
& w_{\varepsilon\varepsilon}(k, \varepsilon, x) \geq 0 \text{ (for all } k \in (0, 1), x \in R_0^+) \\
& \quad \Updownarrow \\
& \quad 2x(1-k)u''[x(1-k)(1+\varepsilon)] \\
& + x^2(1-k)^2(1+\varepsilon)u'''[x(1-k)(1+\varepsilon)] \geq 0 \text{ (for all } k \in (0, 1), \varepsilon \in [-1, \infty)) \\
& \quad \Updownarrow \\
& \quad 2 + X \frac{u'''(X)}{u''(X)} \leq 0 \text{ (for all } X > 0).
\end{aligned}$$

So that $RP \geq 2 \Rightarrow B_3 \succeq A_3$.

(ii) *necessity.* Consider the gambles B_3 and A_3 with $\tilde{\varepsilon}$ being a random variable taking on the values k and $-k$ with equal probability. Let k be a small positive number. Inequality (2) then becomes

$$\begin{aligned}
& \frac{1}{2}u[x(1+k)] + \frac{1}{2}u[x(1-k)] - u[x] \geq \\
& \frac{1}{2}u[x(1-k^2)] + \frac{1}{2}u[x(1-k)^2] - u[x(1-k)]. \tag{3}
\end{aligned}$$

Using a third order Taylor expansion of $u[x(1+k)]$, $u[x(1-k)^2]$ around $k = 0$, inequality (3) can be written as

$$\begin{aligned}
& 2u''(x)x^2k^3 \geq -u'''(x)x^3k^3 + O(k^4) \\
& \quad \Updownarrow \\
& 2 \leq -\frac{u'''(x)x}{u''(x)} + \frac{1}{u''(x)x^2} \frac{O(k^4)}{k^3},
\end{aligned}$$

with $\lim_{k \rightarrow 0} \frac{O(k^4)}{k^3} = 0$. We have established that $B_3 \succeq A_3$ for any pair $k \in (0, 1)$ and any $\tilde{\varepsilon} \in [-1, \infty)$ implies $RP(X) \geq 2$ for any $X > 0$. ■

A comment similar to the one made for risk aversion applies for prudence. When the sure loss and the zero-mean risk are additive, positive prudence ($u''' > 0$) implies a preference for harm disaggregation. However, in the multiplicative case the condition is more demanding: the preference for harm disaggregation requires that prudence be strong enough ($RP \geq 2$).

An interesting interpretation of proposition 2 can again be given through a transformation $h(\cdot)$ ($h' > 0, h'' < 0$ and $h''' > 0^5$) of function $u(x) = \ln(x)$. In an EU framework with a logarithmic utility, $A_3 \sim B_3$ since

$$\frac{1}{2}E \ln(x(1-k)(1+\tilde{\varepsilon})) + \frac{1}{2} \ln(x) = \frac{1}{2} \ln(x(1-k)) + \frac{1}{2}E \ln(x(1+\tilde{\varepsilon})).$$

In this case, the better apportionment in B_3 is exactly compensated for by its higher variance of final wealth. If besides $h' > 0$ and $h'' < 0$, the transformation h also satisfies $h''' > 0$, then RP exceeds 2 (see appendix) and again more weight is attached to the apportionment effect so that $B_3 \succ A_3$.

Before concluding, let us mention that the benchmark values of RRA and RP could also be related to a willingness to trade-off different moments of the lotteries. In the comparison between B_2 and A_2 , final wealth has not only a lower mean in B_2 but also a lower variance. As far as B_3 and A_3 are concerned, these lotteries yield the same expected final wealth but the higher variance of B_3 is compensated for by a lower skewness.⁶

⁵ $h''' > 0$ corresponds to a convexification of h' .

⁶A detailed discussion of these trade-offs between the successive moments can be found

4 Conclusion

The existing literature on savings, insurance and portfolio choices under risk has revealed that quite often comparative statics results depend, among other things, upon the values of the coefficients of relative risk aversion and relative prudence. More specifically the benchmark values of RRA and RP , taken into consideration inside these models, are respectively 1 and 2.

In this paper, we have given a more fundamental interpretation of these benchmark values which is independent of the institutional environment in which the choice is made. This result has been obtained by applying to multiplicative risks the notion of risk apportionment that was used for additive risks in order to justify the alternating signs of successive derivatives of the vNM utility function.

Finally, we believe that the relatively simple nature of the lotteries involved should easily allow for an experimental determination of individual risk attitudes.

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5 Appendix

Let us define function $v(\cdot)$ such as $v(x) = h(\ln x)$ with, for all $y \in R$, $h'(y) > 0$ and $h''(y) < 0$. We then obtain

$$\begin{aligned}
 v'(x) &= h'(\ln x) \times \frac{1}{x}, \text{ and} \\
 v''(x) &= h''(\ln x) \times \frac{1}{x^2} - h'(\ln x) \times \frac{1}{x^2},
 \end{aligned}$$

so that the coefficient of relative risk aversion becomes

$$RRA_v = -\frac{h''(\ln x)}{h'(\ln x)} + 1.$$

Under $h'(y) > 0$ and $h''(y) < 0$, RRA_v necessarily exceeds unity.

Similarly, the coefficient of relative prudence is given by

$$\begin{aligned}
 RP_v &= -x \frac{v'''(x)}{v''(x)} \\
 &= -x \frac{h'''(\ln x) x^{-3} - 3h''(\ln x) x^{-3} + 2h'(\ln x) x^{-3}}{h''(\ln x) x^{-2} - h'(\ln x) x^{-2}} \\
 &= \frac{-h'''(\ln x) + h''(\ln x)}{h''(\ln x) - h'(\ln x)} + 2.
 \end{aligned}$$

Under $h'(y) > 0$, $h''(y) < 0$ and $h'''(y) > 0$, RP_v necessarily exceeds two.